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A note on the spot-forward no-arbitrage relations in an investment-production model for commodities ^{*}

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Abstract

Because of storability constraints, standard no-arbitrage arguments cannot be safely applied in markets of commodities such as energy. In this paper, we propose an alternative approach to justify the convergence of forward towards spot prices as time-to-maturity goes to zero. We show that the classical no-arbitrage relationship between spot and forward prices holds through the well-posedness of an expected profit maximization problem for an agent producing and storing a commodity while trading in forward contracts. A consequence of this is that the forward price of energy can be seen as risk-neutral expectation of the spot price at maturity. Moreover, we obtain an explicit formula for the forward volatility and provide a heuristic analysis of the optimal solution for the production/storage/investment problem in a Markovian setting.

1 Introduction

In this article we aim at explaining the relationship between spot and forward prices of a given commodity, that can be storable or not, within a parsimonious model of commodity prices (spot and forward) as a consequence of the well-posedness of a simple optimisation problem for an agent producing and storing (when possible) a given commodity and, at the same time, trading in the forward market. We stress that the approach used here to get the no-arbitrage forward/spot relation is conceptually different from the one we adopted in previous papers [1, 3] for electricity markets, which consists in enlarging the electricity market by allowing trading in the spot markets of fuels and deducing the spot/forward relation for electricity from the analogue relation assumed to hold in fuel markets. In the rest of this introduction, we will make a short review of the literature related to our main topic as well as a brief description of our results.

It is well-known that commodities futures prices fail to satisfy the no-arbitrage relation that stands for investment assets such as bonds and stocks which can be stored at no cost [21, chap. 4, pp. 140-43]. Precisely, the futures price at time t for delivery at time T of a stock, $F(t, T)$, is the spot price at time t , S_t , capitalised with interest rate r : $F(t, T) = S_t e^{r(T-t)}$. More surprisingly, it has long been acknowledged that the introduction of a storage cost in the former relation is not enough to restore it in the commodity market. Indeed, it is possible to observe inventories storage and futures prices which are backwardated [34]. The no-arbitrage relation that holds for stocks and bonds is thus saved only by the introduction of an extra variable resuming the fact that operators prefer holding the commodity than a futures contract, the convenience yield [23]. With the introduction of this concept, the former relation reads $F(t, T) = S_t e^{(r-y)(T-t)}$, where y is the convenience yield net of the storage cost.

In the economic and financial literature, an important effort has been undertaken to develop models capable of explaining and reflecting the interaction between the spot and the futures prices of a

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commodity through the usage of the storage. Within this huge literature, the closest articles to our concern are [12, 30]. Therein the authors develop a production-storage model implying the connection between spot and futures prices. Nevertheless, those models are mainly developed to allow comparative statics and give little information on the condition under which no-arbitrage holds. On the other hand, pure financial models were designed for pricing commodity derivatives using the notion of convenience yield [32]. But, in these models, the relation between spot and futures relies on the hypothesis that there exists a unique risk-neutral measure to be calibrated to market data.

The development of electricity markets in the last thirty years has challenged the idea that the futures prices of a commodity are linked to its spot price by a convenience yield. Indeed, electricity can not be stored and still, futures prices can exhibit both contango and backwardation structure [9]. Many authors have pointed out the fact that the convenience yield may not apply in the case of electricity and that the no-arbitrage method used in mathematical finance to obtain a risk-neutral measure for pricing could not apply to the case of electricity [7]. Moreover, due to the particular nature of futures contracts available on electricity markets (which are basically swaps, see [8]), even the convergence of the futures price to the spot price is an issue for this commodity [33].

Indeed, since it could not rely on the idea of a convenience yield, the case of electricity has fostered the research on finding a good setting to restore a relation between its spot and futures prices. The first approaches relied on two-date equilibrium models [5, 10, 2] where the risk neutral measure is obtained as a function of the risk aversion parameters of the agents. A more complex approach consists in the extension of the market beyond the spot and the money market to include production factors (fuels) [1, 3] or production constraints [11] or gas storage levels [19]. The spot is deduced as the result of an equilibrium between production and consumption. The futures price is then obtained as an expectation of the spot price.

Our attempt here is not to provide a general theory of the joint dynamics of those two kinds of prices, spot and futures, but to perform an analysis of how the futures can be related to the spot by arbitrage arguments, independently of its storability properties. The no-arbitrage relation is obtained here in a simple model of production and storage of a single operator who is maximising his expected utility. The model is detailed in Section 2. The operator has no impact on the spot price, the storage cost and production functions are supposed convex, the storage capacity is bounded as well as the instantaneous storage or de-storage. Moreover, the operator has access to a forward market with a single maturity. He still has no impact on the forward price and the liquidity is supposed unbounded.

In this setting, the existence of a risk-neutral measure is obtained in Section 3 as a consequence of the finiteness of his value function. Stated differently, this results means that if there were no risk-neutral measures, the operator could take advantage of his production capacity or storage facilities to get an infinite utility. In the same section, we also prove that the futures price always converges to the spot price, regardless its storability property. This point is important especially for electricity prices. Indeed, it has been pointed out that futures prices of electricity has a poor predictive power about realised spot prices [28]. Nevertheless, the former study was done on monthly contracts while the convergence issue concerns only maturities close to zero. And, for instance, day-ahead futures contracts quoted on the German electricity market exhibit lower discrepancy with the realised spot price (see [33]).

In Section 4, we discuss the investment-production problem faced by the operator with a specification of the demand dynamics. We obtain an explicit formula for the volatility of the forward contract and relate it to the volatility of the underlying conditional demand. Moreover, in a Markovian setting and for an agent with a utility function of power type, we argue that the optimal command for the storage management is of a bang-bang type. Unsurprisingly, the decision to store or destore the commodity is based on the comparison between the spot price and the ratio between the marginal utility of one unit of storage and the marginal utility of wealth. Finally, the optimal strategy of selling and buying forward contract is such that the operator holds a long position if the futures prices exhibit a positive trend. We conclude this paper in Section 5.

2 The model for the individual producer

Let $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. (\mathcal{F}_t) is any \mathbb{P} -completed and right-continuous. All processes considered in this paper are assumed to be defined in this space and adapted to this filtration.

Our agent is a producer of commodity, who also acts in the derivative market associated to this commodity. This producer is a price taker; on the physical market, he has the possibility to sell all his production, to store part of it, or to reduce his stocks; on the derivative market, he has the possibility to buy or sell a certain amount of a unique forward contract. These choices depend on the price conditions he faces, both on the physical and on the derivative markets. Let us first focus on the physical market.

2.1 The profit on the physical market

On the physical market, the agent has the possibility to decide how much he produces, and to manage his stocks dynamically. His instantaneous profit π_t can be written:

$$\pi_t = (q_t - u_t)S_t - c(q_t) - k(X_t),$$

where:

- q_t is the production of the agent,
- u_t is the amount stored or destored,
- S_t is the spot price of the commodity,
- $c : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the production function,
- $k : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the storage function, with $k(0) = 0$,
- X_t is the storage level at time t .

Assumption 2.1. *We assume that both functions c and k are differentiable, strictly increasing, strictly concave and nonnegative.*

Moreover, we will always work under this standing assumption on the spot dynamics:

Assumption 2.2. *Let (S_t) be a bounded continuous process.*

The constraints faced by the producer on the physical market can be summarised as follows:

- the agent production cannot exceed his capacity: $q_t \in [0, \bar{q}]$ for some $\bar{q} > 0$;
- instantaneous storage and de-storage are bounded, with $u_t \in [\underline{u}, \bar{u}]$ for given thresholds $\underline{u} < 0 < \bar{u}$;
- the storage capacity itself is bounded: $X_t \in [0, \bar{X}]$, with the additional positivity constraint on inventories: $X_t \geq 0$ a.s. for all $t \in [0, T]$;
- the storage dynamics is : $dX_t = u_t dt$, with $X_0 = u_0 > 0$.

In this setting, there is no uncertainty on production. As we will see below, the only uncertainty source comes from the demand side. This hypothesis is made to simplify the computation and the formulas. The introduction of a stochastic production capacity driven by a Brownian motion independent of the one driving the demand would introduce market incompleteness.

2.2 The investment in the derivative market

We assume, for the sake of simplicity, that the interest rate is zero so futures and forward contracts are equivalent. In our derivative market model, there is only one futures contract available, $F_t = F_t(T)$, for a given maturity $T > 0$. We assume that the forward price process (F_t) is a continuous semimartingale, adapted to the filtration (\mathcal{F}_t) . The investment portfolio in this contract is given by

$$V_T^\theta = \int_0^T \theta_t dF_t, \quad (2.1)$$

where θ is a real-valued predictable (F_t) -integrable process. We make the implicit hypothesis that the futures market is liquid, which is standard in this context. This is the case for short-term maturities of storable commodities (oil, gas). This is less true for electricity markets [4, Chap. 2, Sec. 2.2.3].

2.3 The production-investment problem

As a producer of commodity, the agent acts so as to maximize the expected utility of his terminal wealth. His utility function $U : \mathbb{R}_+ \rightarrow [-\infty, \infty[$ satisfies Inada conditions and is such that $U(x) \rightarrow \infty$ whenever $x \rightarrow \infty$. Moreover, we assume RAE (Reasonable Asymptotic Elasticity [24]):

$$AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

In this setting, we thus consider the following production-investment problem:

$$v(r_0) := \sup_{u, q, \theta} \mathbb{E} \left[U \left(r_0 + \int_0^T \pi_t dt + V_T^\theta + \theta_{T-}(F_T - S_T) \right) \right], \quad (2.2)$$

where:

- $r_0 > 0$ is the initial wealth;
- π_t is the instantaneous profit on the physical market;
- V_T^θ is the investment portfolio in the forward market as in (2.1);
- the term $\theta_{T-}(F_T - S_T)$ can be explained by delivery conditions (see the heuristic discussion in discrete time below).

The controls (u, q, θ) have to satisfy the following additional constraints:

- Constraint on the wealth of the agent to prevent infinite borrowing:

$$R_t^{x, q, u, \theta} := r_0 + \int_0^t \pi_s ds + V_t^\theta + \theta_{T-}(F_T - S_T) \mathbf{1}_{(t=T)} \geq 0, \quad t \in [0, T]. \quad (2.3)$$

- The production-storage controls $(u_t, q_t)_{t \in [0, T]}$ are adapted processes with respect to the filtration (\mathcal{F}_t) and they satisfy the constraints previously described in Section 2.1.

Discrete-time heuristics. We provide here some heuristics in discrete-time to justify the form of the continuous-time problem (2.2). The terminal wealth for an investor-producer who produces, e.g., energy out of fuels and trade in forward contracts on energy over the finite time grid $\{0, 1, \dots, T\}$ with $T \in \mathbb{N}$ can be written as follows:

$$\begin{aligned} \sum_0^{T-1} [(q_t - u_t)S_t - c(q_t) - k(X_t)] &+ \sum_0^{T-1} \theta_t (F_{t+1} - F_t) \\ &+ [\theta_{T-1}F_{T-1} - h_T S_T - c(q_T) - k(X_T)], \end{aligned}$$

where h_T is the quantity bought or sold at terminal date that allows to satisfy the delivery condition

$$\theta_{T-1}(F_{T-1} - S_T) + (q_T - u_T)S_T - c(q_T) - k(X_T),$$

so that the terminal total wealth becomes

$$\sum_0^T [(q_t - u_t)S_t - c(q_t) - k(X_t)] + \sum_0^{T-2} \theta_t(F_{t+1} - F_t) + \theta_{T-1}(F_T - S_T), \quad (2.4)$$

which constitutes the discrete-time analogue of the total wealth appearing inside the utility function in (2.2).

Among other things, we will prove in the next section that when the problem is well-posed one must have $F_T = S_T$, so that the third term in (2.4) (as well as its continuous-time counterpart in (2.2)) will vanish.

Our first objective is to deduce from the well-posedness of problem (2.2) the no-arbitrage condition that would link spot and forward prices, i.e. to prove that there exists some equivalent probability measure Q such that

$$F_t = \mathbb{E}_Q[S_T | \mathcal{F}_t], \quad t \in [0, T],$$

and try to compute it explicitly.

3 Existence of the optimum and spot-forward no-arbitrage relations

In this section we derive existence and uniqueness of an optimal solution (q^*, u^*, θ^*) for the optimization problem (2.2) and at the same time we obtain no-arbitrage relations between spot and forward prices as well as the convergence of forward prices to spot prices when time-to-maturity goes to zero. Moreover, we will see that the optimal production q^* can be computed explicitly even in this quite general framework. For computing explicitly (u^*, θ^*) we have to add more structure to our market model. This will be discussed in the next section.

Assumption 3.1. *Let $v(r_0) < \infty$ for some initial wealth $r_0 > 0$.*

Convergence of forward to spot prices and no-arbitrage relationship. Our first result states that as long as our optimization problem is well-posed, one must have convergence of forward prices towards the spot price as time-to-maturity tends to zero.

Proposition 3.2. *Under all our assumptions, we have $F_T = S_T$.*

Proof. Assume that $\mathbb{P}(F_T \neq S_T) > 0$ and let $A = \{F_T > S_T\}$ and $B = \{F_T < S_T\}$. Consider the following production-investment strategy

$$q = u = 0, \quad \theta_t := (\alpha \mathbf{1}_A - \beta \mathbf{1}_B) \mathbf{1}_{t=T},$$

where α, β are arbitrary positive numbers. Since A and B are \mathcal{F}_{T-} -measurable (S_t and F_t are both continuous processes), θ is clearly a predictable and F_t -integrable investment strategy. Pursuing such a strategy gives terminal wealth

$$x + \alpha(F_T - S_T) \mathbf{1}_A + \beta(S_T - F_T) \mathbf{1}_B,$$

so that letting $\alpha \rightarrow \infty$ and $\beta = 0$ if $\mathbb{P}(A) > 0$ or $\beta \rightarrow \infty$ and $\alpha = 0$ if $\mathbb{P}(B) > 0$ we get $v(x) = \infty$ (recall that $U(x) \rightarrow \infty$ when $x \rightarrow \infty$), which contradicts the well-posedness of our maximization problem. Thus, we can conclude that a.s. $F_T = S_T$. \square

Now, before proving the existence of a solution to our optimization problem, we turn to deducing the no-arbitrage property for forward contracts from the finiteness of $v(r_0)$. This is the content of the next proposition, that adapts arguments from Proposition 1.2 in Ankirchner and Imkeller [6]. We are going to deduce something stronger, namely NFLVR not only for the investment part but also for production and storage. To be more precise, let us reformulate NFLVR for our setting.

Definition 3.3. A free lunch with vanishing risk with production and storage is a sequence of admissible plans $(q_t^n, u_t^n, \theta_t^n)$, $n \geq 1$, such that $R_T^{0,n} := R_T^{0,q^n, u^n, \theta^n}$ converges a.s. towards some nonnegative r.v. R_T^0 satisfying $\mathbb{P}(R_T^0 > 0) > 0$ and $\|(R_T^n)^-\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. We will say that NFLVR with production and storage is satisfied if there are no such strategies in the model.

Notice that since the production and storage controls are bounded, there exists a constant $M > 0$ such that $|\int_0^T \pi_t dt| \leq M$ for any admissible (q, u) giving the instantaneous profit π_t . This fact will be used in the proof of the following result.

Proposition 3.4. Under all our assumptions and for all $r_0 > M$ we have that $v(r_0) < \infty$ implies NFLVR with production and storage. In particular, NFLVR for forward prices (F_t) also holds, which in turn implies that there exists at least one equivalent probability measure Q under which F_t is a local martingale.

Proof. Suppose that NFLVR with production and storage is violated, so that we can find a sequence of terminal payoffs $R_T^n = \int_0^T \pi_t^n dt + \int_0^T \theta_t^n dF_t$ such that $R_T^n \rightarrow R_T^0$ for some nonnegative r.v. R_T^0 with $\mathbb{P}(R_T^0 > 0) > 0$ and $\varepsilon_n := \|(R_T^n)^-\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Here π^n denotes the instantaneous profit coming from a production q^n and storage u^n . Now, take any $r_0 > 0$. There exists a $\delta > 0$ such that $\kappa := U(r_0 - M - \delta) \wedge 0 > -\infty$. Now, consider the sequence of strategies $(\tilde{q}^n, \tilde{u}^n, \tilde{\theta}^n)$ where

$$\tilde{q}^n = \frac{\delta}{\varepsilon_n} q^n \wedge \bar{q}, \quad \tilde{u}^n = \frac{\delta}{\varepsilon_n} u^n \wedge \bar{u} \vee (-\bar{u}), \quad \tilde{\theta}^n = \frac{\delta}{\varepsilon_n} \theta^n,$$

and denotes \tilde{R}^n the corresponding payoff. Clearly, $\tilde{R}_T^n \geq -\delta$ and moreover the r.v.s $U(r_0 + \tilde{R}_T^n)$ is bounded from below by κ . Indeed

$$U\left(r_0 + \int_0^T \tilde{\pi}_t^n dt + V_T^{\tilde{\theta}^n}\right) = U\left(r_0 + \int_0^T \tilde{\pi}_t^n dt + \frac{\delta}{\varepsilon_n} V_T^{\theta^n}\right) \geq U\left(r_0 - M + \frac{\delta}{\varepsilon_n}(-\varepsilon_n)\right) = \kappa > -\infty.$$

Since R_T^n converges to the nontrivial nonnegative r.v. R_T^0 and any profit is bounded by a constant M , one can find an integer n_0 and real numbers $b, c > 0$ such that $\mathbb{P}(V_T^{\theta^n} > b) > c$ for all $n \geq n_0$. Since $\kappa \leq 0$ we have

$$\begin{aligned} \liminf_n \mathbb{E}\left[U\left(r_0 + \int_0^T \tilde{\pi}_t^n dt + V_T^{\tilde{\theta}^n}\right)\right] &= \liminf_n \mathbb{E}\left[U\left(r_0 + \int_0^T \tilde{\pi}_t^n dt + \frac{\delta}{\varepsilon_n} V_T^{\theta^n}\right)\right] \\ &\geq \liminf_n \mathbb{E}\left[\kappa \mathbf{1}_{\{V_T^n \leq b\}} + U\left(r_0 + \int_0^T \tilde{\pi}_t^n dt + \frac{\delta}{\varepsilon_n} b\right) \mathbf{1}_{\{V_T^n > b\}}\right] \\ &\geq \liminf_n \mathbb{E}\left[\kappa(1-c) + U\left(r_0 - M + \frac{\delta}{\varepsilon_n} b\right) c\right] = \infty, \end{aligned}$$

yielding that $v(r_0) = \infty$. To conclude, just recall that NFLVR with production and storage clearly implies NFLVR for trading in forward contracts, which yields in turn the existence of at least one equivalent probability measure Q for which F_t is a local martingale (see the seminal paper [18]). \square

Existence and separation principle. An immediate consequence of the previous convergence result is the following separation principle, stating that solving our optimization problem is equivalent to maximize first with respect to production control q and then with respect to the storage/investment controls (u, θ) . On the other hand, maximizing the production can also be performed in two steps. Let us denote

$$v(r_0) = \sup_{u, q, \theta} \mathbb{E}\left[U\left(r_0 + Y_T^q + Z_T^u + V_T^\theta\right)\right],$$

where we set

$$Y_T^q := \int_0^T (q_t S_t - c(q_t)) dt, \quad Z_T^u := - \int_0^T (u_t S_t + k(X_t)) dt$$

We can solve our problem in two separate steps: first solve $v(r_0)$ with respect to production control q (for given u, θ), second with respect to the controls (u, θ) . Let us start from the production side.

Proposition 3.5. *Under our assumptions, for any given admissible investment strategy θ and storage policy u , the optimal production controls q^* is given by*

$$q_t^* = (c')^{-1}(S_t) \mathbf{1}_{\{(c')^{-1}(S_t) \leq \bar{q}\}}, \quad t \in [0, T]. \quad (3.1)$$

Proof. It suffices to maximize ω -wise inside the integral in the term $Y_T^{q,u}$ containing the production controls. Differentiate with respect to q_t gives $S_t - c'(q_t) = 0$ so that, taking into account the constraints $q_t \in [0, \bar{q}]$ and since c is strictly concave, we have

$$q_t^* = (c')^{-1}(S_t) \mathbf{1}_{\{(c')^{-1}(S_t) \leq \bar{q}\}}.$$

The proof is completed. \square

Let us denote

$$Y_T^* := Y_T^{q^*} = \int_0^T (q_t^* S_t - c(q_t^*)) dt$$

where q_t^* is given by (3.1). Now, let us consider the optimal storage/investment problem

$$v(r_0) := \sup_{u, \theta} \mathbb{E} [U(r_0 + Y_T^* + Z_T^u + V_T^\theta)]. \quad (3.2)$$

The next result establishes existence of a unique optimal storage/investment policy (u^*, θ^*) .

Proposition 3.6. *Under all our assumptions, there exists a unique solution (u^*, θ^*) to the problem (3.2).*

Proof. First of all, if one admits existence of a solution, its uniqueness follows at once from the strict concavity of the utility function U . Let (u^n, θ^n) be a maximizing admissible sequence for the problem

$$v(r_0) := \sup_{u, \theta} \mathbb{E} [U(r_0 + Y_T^* + Z_T^u + V_T^\theta)],$$

i.e. $\mathbb{E}[U(r_0 + Y_T^* + Z_T^n + V_T^n)] \rightarrow v(r_0)$ as $n \rightarrow \infty$, where we denoted

$$Z_T^n := - \int_0^T (u_t^n S_t + k(X_t^n)) dt, \quad X_t^n := u_0 + \int_0^t u_s^n ds, \quad V_T^n := \int_0^T \theta^n dF_t.$$

We are going to prove compactness property of the sequences u^n and θ^n separately. For the sequence of storage strategies u^n , we will use Komlós theorem, stating that for any sequence of r.v.'s (ξ^n) bounded in L^1 one can extract a subsequence (ξ^{n_k}) converging a.s. in Cesaro sense to a random variable $\xi^0 \in L^1$ (see, e.g., Theorem 5.2 in [22]). We apply this theorem to the sequence of processes u^n , that can be viewed as random variables defined on the product space $(\Omega \times [0, T], \mathcal{P}, d\mathbb{P}dt)$ where \mathcal{P} is the predictable σ -field. The sequence u^n is clearly in L^1 since it takes values in the interval $[-\bar{u}, \bar{u}]$. Thus, there exists a predictable process u^0 taking values in the same interval, such that the Cesaro mean sequence $\tilde{u}^n := (1/n) \sum_{j=1}^n u^j$ converges a.e. towards u^0 . Indeed it's immediate to check that the sequence \tilde{u}^n takes values in $[-\bar{u}, \bar{u}]$ as well. Moreover, the cumulated storage process along the new sequence, $\tilde{X}_t^n := \int_0^t \tilde{u}_s^n ds$, is well-defined since each \tilde{u}^n is bounded and it takes values in $[0, \bar{X}]$. By Lebesgue dominated convergence we have that $\tilde{X}_t^n \rightarrow X_t^0 := \int_0^t u_s^0 ds$ a.s. for all $t \in [0, T]$. Since the function k is continuous, we have $k(\tilde{X}_t^n) \rightarrow k(X_t^0)$ a.s. for all t . Finally, thanks once more to boundedness of the controls and continuity of k , the r.v. \tilde{Z}_T^n is bounded by some constant times $\int_0^T S_t dt$ which is bounded (since S_t is bounded uniformly in t), so that applying dominated convergence once more we get $\tilde{Z}_T^n \rightarrow Z_T^0$ a.s. as $n \rightarrow \infty$.

As for the compactness of the sequence of investment strategies θ^n , we can work with the corresponding wealth processes Cesaro mean sequence, that we denote \tilde{V}_T^n . The admissibility property and the uniform boundedness of \tilde{Z}_T^n yields that this sequence is uniformly bounded from below by some constant. Therefore, we can apply the Delbaen-Schachermayer lemma (see, e.g., [18, Lemma 9.8.1]), implying that there exists a convex combination $\hat{V}_T^n \in \text{conv}(\tilde{V}_T^n, \tilde{V}_T^{n+1}, \dots)$, which converges a.s. to some V_T^0 . It is standard to prove that $V_T^0 = \int_0^T \theta_t^0 dF_t$ for some admissible θ^0 . Moreover,

applying this procedure to the Cesaro means of storage strategies \tilde{u}^n gets another sequence of admissible storage strategies \hat{u}^n converging a.s. to the same process as before u^0 . To sum up, we have just found a sequence (\hat{u}^n, \hat{V}_T^n) converging a.s. to some (u^0, V_T^0) .

To conclude the proof, we need to show that

$$v(r_0) = \mathbb{E}[U(x + Y_T^* + V_T^0 + Z_T^0)].$$

To do so, it suffices to use the assumption that U satisfies RAE by proceeding as in the proof of, e.g., Theorem 7.3.4 in [27]. Repeating his arguments gives us the inequality above getting that (u^0, θ^0) is the optimal storage control (u^*, θ^*) . The proof of existence is now completed. \square

Remark 3.7. The fact that the investment-production problem above can be solved in successive steps does not mean that the optimal controls are independent. Only the production control q can be deduced independently from u and θ . Indeed, since the producer has no impact on the spot price, his optimal strategy is simply to equal his marginal cost of production with the spot price. Thus, whether there exists or not a futures market, one would observe the same production level q . This is not the case for the optimal storage policy u and the optimal investment strategy θ . They are not independent. Thus, the introduction of a futures market should modify the way the storage facilities are managed.

Our previous results implies in particular that F_t is a local Q -martingale under some equivalent probability measure Q with terminal value at time T given by the spot price S_T . If F_t was a true martingale under Q , we would recover the well-known relation

$$F_t = \mathbb{E}^Q[S_T | \mathcal{F}_t], \quad t \in [0, T].$$

This would be the case when, e.g., F_t is an Itô process whose Brownian part have a square-integrable volatility process as in the next section, or more generally when F_t is of class (D) (see, e.g. [29, III.3]).

4 The optimal investment-production problem

The spot price S_t results from the availability of the commodity. This availability is measured through the confrontation of the total capacities of the market and the demand for the commodity, in the following way:

$$S_t = b \cdot g(\bar{C} - D_t) \cdot f(D_t), \quad (4.1)$$

with b a constant of normalization for dimension purposes, $\bar{C} > 0$ the maximum available production and storage capacities of the market (supposed constant), D_t the total exogenous demand for the commodity (which is an (\mathcal{F}_t) -adapted continuous process) and $f(D)$ the marginal cost of production and storage for a demand level D .

As there is a non negativity constraint on inventories, the spot price can jump to very high levels when the total capacities are not sufficient to fully satisfy the demand. This behavior is captured by the scarcity function g :

$$g(x) = \mathbf{1}_{x>0} \cdot \min(1/x, 1/\epsilon) + \mathbf{1}_{x<0} 1/\epsilon.$$

The effect of scarcity on commodity prices is clearly illustrated in oil in [13, p. 56, fig. 10]. The specific form above has been successfully implemented in the case of electricity spot prices in [3].

The production optimization problem being solved (Proposition 3.5), it remains to maximize

$$v(x) = \sup_{u, \theta} \mathbb{E} [U(x + Y_T^* + Z_T^u + V_T^\theta)],$$

where

$$Y_T^* = \int_0^T (q_t^* S_t - c(q_t^*)) dt \quad \text{with} \quad q_t^* = (c')^{-1}(S_t) \mathbf{1}_{\{(c')^{-1}(S_t) \leq \bar{q}\}}, \quad t \in [0, T].$$

We recall that $Z_T^u = u_0 + \int_0^T u_t dt$ is the cumulated storage and $V_T^\theta = \int_0^T \theta_t dF_t$ the invested portfolio over the period $[0, T]$.

We assume that the forward price process (F_t) is an Itô process (or a diffusion to exploit Markov property). More precisely:

Assumption 4.1. 1. Let the demand for energy D_t be mean reverting, with a long-run mean set to zero:

$$dD_t = aD_t dt + \sigma dW_t, \quad (4.2)$$

where a, σ are constants and W is a standard Brownian motion. We denote (\mathcal{F}_t) the natural filtration generated by it and completed with the \mathbb{P} -null sets.

2. Let assume

$$dF_t = \alpha_t dt + \beta_t dW_t,$$

where α, β are (\mathcal{F}_t) -predictable real-valued processes such that a.s.

$$\int_0^T |\alpha_t| dt + \mathbb{E} \left[\int_0^T \beta_t^2 dt \right] < \infty.$$

The integrability assumption on the volatilities is here only to have the (true) martingale property of F_t and consequently the very useful formula $F_t = \mathbb{E}^Q[S_T | \mathcal{F}_t]$.

4.1 Equivalent martingale measures and forward volatility

Under the assumptions of the previous sections and Assumptions 4.1, we can deduce that

$$F_t = \mathbb{E}^Q[S_T | \mathcal{F}_t],$$

where Q is an equivalent martingale measure for the forward process (F_t) , which means that Q must necessarily satisfy

$$L_t^\lambda := \frac{dQ}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left\{ \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right\},$$

where λ is a (\mathcal{F}_t) -adapted process (viewed as “market price of demand risk”) such that

- $\alpha_t - \lambda_t \beta_t = 0$ a.e. $d\mathbb{P} \otimes dt$,
- $\int_0^T \lambda_s^2 ds < \infty$,
- $\mathbb{E}[L_T^\lambda] = 1$.

At this point, in order to specify completely the dynamics under \mathbb{P} of the forward price, we need to assume a particular and tractable form for the market price of demand risk λ_t .

Assumption 4.2. Let us assume that $\lambda_t = \lambda_0(t) + \lambda_1(t)D_t$, $t \in [0, T]$, where $\lambda_0, \lambda_1 : [0, T] \rightarrow \mathbb{R}$ are deterministic functions such that the last three properties above are satisfied.

A consequence of this assumption is that the drift of forward prices α_t takes the form $\alpha_t = (\lambda_0(t) + \lambda_1(t)D_t)\beta_t$, which is completely determined up to the volatility β_t .

We will see in a moment that the special form of the production function defining the spot price S_T in (4.1) implies a particular functional form for the volatility of the forward price process (F_t) .

Let Q be the equivalent martingale measure corresponding to the market price of demand risk λ_t as in Assumption 4.2. Under such a measure, the demand has dynamics

$$dD_t = ((a + \lambda_1(t))D_t + \lambda_0(t))dt + \sigma dW_t^Q,$$

where W^Q is a standard Q -BM. Thus, the conditional distribution of D_T given D_t under Q is Gaussian with conditional mean $m_{t,T}^Q$ and variance $\Sigma_{t,T}^2$ given by

$$m_{t,T}^Q = e^{\int_t^T (a + \lambda_1(s)) ds} \left(D_t + \int_t^T e^{-\int_0^s (a + \lambda_1(u)) du} \lambda_0(s) ds \right), \quad \Sigma_{t,T}^2 = \sigma^2 \int_t^T e^{-2 \int_t^s (a + \lambda_1(u)) du} ds. \quad (4.3)$$

Assume for the sake of simplicity that $\kappa = 1$. Moreover, we choose the marginal cost of production f to be equal to

$$f(d) = d^\alpha \mathbf{1}_{(0 \leq d \leq M)} + M^\alpha \mathbf{1}_{(d \geq M)}, \quad d \in \mathbb{R},$$

for some exponent $\alpha \in (0, 1)$ and some upper bound $M > 0$ such that our conditions on f are fulfilled. Moreover, we assume that $M \geq \bar{C} - \epsilon$.

Under all these assumptions, we can express the spot price S_t as a function of the demand $S_t = \psi(D_t)$, where the function ψ is given as follows

$$\psi(d) = b \cdot \left(\frac{d^\alpha}{\epsilon} \mathbf{1}_{(0 \leq d < \bar{C} - \epsilon)} + \frac{d^\alpha}{\bar{C} - d} \mathbf{1}_{(\bar{C} - \epsilon \leq d < M)} + \frac{M}{\bar{C} - d} \mathbf{1}_{(d \geq M)} \right). \quad (4.4)$$

Notice that the spot price S_t is always nonnegative.

The forward price at time t computed under the above measure Q is given by

$$F_t = E_t^Q[\psi(D_T)], \quad t \in [0, T],$$

where E_t^Q denotes the conditional Q -expectation given $\mathcal{F}_t = \mathcal{F}_t^D$. We denote $h_{T, D_t}(y)$ the conditional density of D_T given D_t that is

$$h_{T, D_t}(y) = \frac{1}{\Sigma_{t, T} \sqrt{2\pi}} \exp\left(-\frac{(y - m_{t, T}^Q)^2}{2\Sigma_{t, T}^2}\right),$$

where the mean $m_{t, T}^Q$ and the variance $\Sigma_{t, T}^2$ are given in (4.3). We recall that the variance does not depend on D_t .

We can express the forward price F_t as a function of the demand at time t , D_t , as

$$F_t = \varphi(t, D_t) = \int_{\mathbb{R}} \psi(y) h_{T, D_t}(y) dy.$$

A simple application of Itô's formula together with the martingale property of the forward price F_t under Q gives that the volatility of the forward price, $\beta(t, D_t)$, is given by

$$\beta_t = \beta_t^T = \sigma \frac{\partial \varphi}{\partial d}(t, D_t).$$

If we compute explicitly the first and second derivatives of the forward price, $\varphi(t, D_t)$, with respect to the demand, we obtain the following result giving a complete specification of the parameters of the forward dynamics.

Proposition 4.3. *Under Assumptions 4.1 and 4.2, the well-posedness of our optimal production-investment problem implies that*

$$dF_t = \alpha_t dt + \beta_t dW_t$$

where

$$\begin{aligned} \alpha_t &= \tilde{\alpha}(t, D_t) = (\lambda_0(t) + \lambda_1(t)D_t)\beta_t, \\ \beta_t &= \tilde{\beta}(t, D_t) = 2\sigma \int_{\mathbb{R}} \psi(y)(y - m_{t, T}^Q) e^{\int_0^T (a + \lambda_1(u)) du} h_{T, D_t}(y) dy, \end{aligned}$$

for all $t \in [0, T]$. Moreover, the forward volatility $\tilde{\beta}(t, D_t)$ is increasing in the demand.

4.2 The investment-production optimization problem in a Markovian model

In this section, we provide for the sake of completeness an informal discussion of the optimal solutions within the Markovian model determined in the previous proposition. Let us assume that the preferences of the agent are of the power type, i.e. $U(x) = x^\gamma$, $x > 0$, where $\gamma < 1$. Recall that the problem we want to solve is the following:

$$v(x) := \sup_{(u, q, \theta) \in \mathcal{A}} \mathbb{E} \left[\left(r_0 + \int_0^T \pi_t dt + V_T^\theta \right)^\gamma \right], \quad (4.5)$$

where $r_0 > 0$ is the initial wealth, π_t is the profit rate and is given by

$$\pi_t = (q_t - u_t)S_t - c(q_t) - k(X_t), \quad X_t = u_0 + \int_0^t u_s ds,$$

while $V_T^\theta = \int_0^T \theta_t dF_t$ is the gain coming from investing in a self-financing manner in the forward market. \mathcal{A} denotes the set of all admissible controls (u, q, θ) . More precisely, we will say that a triplet (u, q, θ) is an admissible control if

- $q = (q_t)_{t \in [0, T]}$ and $u = (u_t)_{t \in [0, T]}$ are adapted processes with values, respectively, in $[0, \bar{q}]$ and $[\underline{u}, \bar{u}]$;
- $\theta = (\theta_t)_{t \in [0, T]}$ is any predictable real-valued F -integrable process such that the resulting wealth is a.s. nonnegative at any time, i.e.

$$r_0 + \int_0^t \pi_s ds + V_t^\theta \geq 0, \quad t \in [0, T].$$

The relevant state variable of the problem is $Z = (R, X, D)$ where R is the wealth of the agent, i.e.

$$R_t = r_0 + \int_0^t \pi_s ds + V_t^\theta, \quad t \in [0, T].$$

The dynamics of the state variable is given by

$$\begin{aligned} dR_t &= [(q_t - u_t)\psi(D_t) - c(q_t) - k(X_t) + \alpha(t, D_t)\theta_t] dt + \beta(t, D_t)\theta_t dW_t. \\ dX_t &= u_t dt, \\ dD_t &= aD_t dt + \sigma dW_t. \end{aligned}$$

Let us introduce the value function of the optimization problem as

$$v(t, r, x, d) = \sup_{(u, q, \theta) \in \mathcal{A}_t} \mathbb{E}[(R_T)^\gamma | Z_t = (r, d, x)]$$

where \mathcal{A}_t denotes the set of all admissible controls starting at time t . The corresponding HJB equation is given by

$$-v_t - \sup_{(u, q, \theta) \in A} \mathcal{L}v = 0, \quad \text{with } A := [\underline{u}, \bar{u}] \times [0, \bar{q}] \times \mathbb{R}, \quad (4.6)$$

with terminal condition

$$v(T, x, d, r) = r^\gamma, \quad (4.7)$$

and where

$$\mathcal{L}v = uv_x + adv_d + [(q - u)\psi(d) - c(q) - k(x) + \alpha(t, d)\theta] v_r + \frac{1}{2}\sigma^2 v_{dd} + \frac{1}{2}\beta(t, d)^2 \theta^2 v_{rr}.$$

The HJB equation rewrites :

$$\begin{aligned} 0 &= -v_t - adv_d + k(x)v_r - \frac{1}{2}\sigma^2 v_{dd} \\ &\quad - \sup_{(u, q, \theta) \in A} \left\{ uv_x + [(q - u)\psi(d) - c(q) + \alpha(t, d)\theta] v_r + \frac{1}{2}\beta(t, d)^2 \theta^2 v_{rr} \right\}. \end{aligned}$$

Rearranging the terms gives

$$\begin{aligned} 0 &= -v_t - adv_d + k(x)v_r - \frac{1}{2}\sigma^2 v_{dd} \\ &\quad - \sup_{(u, q, \theta) \in A} \left\{ (v_x - \psi(d)v_r)u - (c(q) - q\psi(d))v_r + \alpha(t, d)\theta v_r + \frac{1}{2}\beta(t, d)^2 \theta^2 v_{rr} \right\}. \end{aligned} \quad (4.8)$$

We notice immediately from the above HJB equation that the optimal candidate rule for the storage management is of a bang-bang policy type. More precisely, we expect to have

$$u_t^* = \underline{u} \mathbf{1}_{(\psi(D_t)v_r > v_x)} + \bar{u} \mathbf{1}_{(\psi(D_t)v_r \leq v_x)}, \quad (4.9)$$

where, to simplify the notation, we dropped the arguments (t, R_t, X_t, D_t) from the derivatives of the value functions v_r and v_x . Depending on the ratio $\eta = v_x/\psi(d)$ between the marginal utility of one unit of storage and the spot price, it is optimal either to buy and store at maximum capacity or to destore and sell at maximum capacity. One may have thought that this ratio should simply be compared to one. This is not the case. It has to be compared to the marginal utility of one unit of wealth, v_R .

We recover the fact that the optimal control for production is to produce until the marginal cost of production equals the spot price, i.e.

$$q_t^* = (c')^{-1}(S_t) \mathbf{1}_{\{(c')^{-1}(S_t) \leq \bar{q}\}}. \quad (4.10)$$

Finally, the optimal control for the trading portfolio is given by :

$$\theta_t^* = -\frac{\alpha(t, D_t)}{\beta(t, D_t)^2} \frac{v_r}{v_{rr}}(t, R_t, X_t, D_t). \quad (4.11)$$

Since it is likely that v_{R^2} is negative because U is concave, one recovers the expected result that the agent holds a long position if futures prices are exhibiting a positive trend. Moreover, θ^* is similar to Sharpe ratio, a tradeoff between the expected trend of the futures prices compared to their volatility.

Remark 4.4. To rigorously solve the optimisation problem, one should prove that the Cauchy problem (4.6, 4.7) admits a unique solution with the required regularity together with a verification theorem. This could be achieved using the techniques developed in, e.g., [26] for quite a large class of multi-dimensional stochastic volatility models. However adapting such a method to our setting could be very technical and would go far beyond the scope of this paper. Therefore we decided not to pursue this task.

5 Conclusion

We developed here a parsimonious model of commodity prices (spot and forward) that can explain the relationship between spot and forward prices based on arbitrage arguments. The argument is not based on the storability of the commodity. The existence of a risk-neutral measure is shown to be a consequence of the finiteness of the value function of an agent maximizing its utility. We also proved that the futures price always converges to the spot price, whatever its storability property. Finally, we briefly discussed the solution of the investment-production problem faced by the agent and showed how the different controls can be separated. In particular, the optimal storage policy is impacted by the possibility of acting on the futures market. This point will be tested in future works.

References

- [1] R. Aïd, L. Campi, A. Nguyen Huu, and N. Touzi. A structural risk neutral model of electricity price. *Int. Jour. of Theoretical and Applied Finance*, 12(7):925–947, 2009.
- [2] R. Aïd, G. Chemla, A. Porchet, and N. Touzi. Hedging and vertical integration in electricity markets. *Management Science*, 57(8):1438–1452, 2011.
- [3] R. Aïd, L. Campi, and N. Langrené. A structural risk neutral model for pricing and hedging power derivatives. *Mathematical Finance*, 23(3):387–438, 2013.
- [4] R. Aïd. Electricity derivatives. *SpringerBriefs in Quantitative Finance*, 2015.

-
- [5] E. J. Anderson and X. Hu. Forward contracts and market power in an electricity market. *Int. J. of Industrial Organization*, 26:679–694, 2008.
- [6] S. Ankirchner and P. Imkeller. Finite utility on financial markets with asymmetric information and structure properties of the price dynamics. *Annals of Henri Poincaré Institute*, (41):479–503, 2005.
- [7] F. E. Benth, L. Ekeland, R. Hauge, and B. F. Nielsen. A note on arbitrage-free pricing of forward contracts in energy markets. *Applied Mathematical Finance*, 10(4):325–336, 2003.
- [8] D. Frestad, F. E. Benth, and S. Koekebakker. Modeling term structure dynamics in the Nordic electricity swap market. *The Energy Journal*, 31(2):53–86, 2010.
- [9] F. E. Benth, R. Biegler-König, and R. Kiesel. An empirical study of the information premium on electricity markets. *Energy Economics*, 36:55 – 77, 2013.
- [10] H. Bessembinder and M. L. Lemmon. Equilibrium pricing and optimal hedging in electricity forward markets. *Journal of Finance*, 23:1347–82, 2002.
- [11] B. Bouchard and A. Nguyen Huu. No marginal arbitrage of the second kind for high production regimes in discrete time production-investment models with proportional transaction costs. *Mathematical Finance*, 23(2), 366–386, 2013.
- [12] M. J. Brennan. The supply of storage. *The American Economic Review*, 48(1):pp. 50–72, 1958.
- [13] B. Büyüksahin, M. S. Haigh, J. H. Harris, J. A. Overdahl, and M. A. Robe. Fundamentals, trader activity and derivative pricing. Available at SSRN: <http://ssrn.com/abstract=966692>. 2008
- [14] M. J. Chambers and R. E. Bailey. A theory of commodity price fluctuations. *Journal of Political Economy*, 104(5):924 – 957, 1996.
- [15] J.-P. Danthine. Martingale, market efficiency and commodity prices. *European Economic Review*, 10(1):1 – 17, 1977.
- [16] A. Deaton and G. Laroque. On the behaviour of commodity prices. *Review of Economic Studies*, 59(1):1 – 23, 1992.
- [17] A. Deaton and G. Laroque. Competitive storage and commodity price dynamics. *Journal of Political Economy*, 104(5):896 – 923, 1996.
- [18] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* 300.1 (1994): 463 – 520.
- [19] S. Douglas and J. Popova. Storage and the electricity forward premium. *Energy Economics*, 30(4):1712 – 1727, 2008.
- [20] I. Ekeland, D. Lautier and B. Villeneuve. A Simple Equilibrium Model for a Commodity Market with Spot and Futures Trades. *FiME Research Paper*, RR-FiME-13-05, 2013.
- [21] A. Eydeland and K. Wolyniec. *Energy and Power Risk Management: New Developments in Modeling, Pricing and Hedging*. Wiley, 2002.
- [22] Y. Kabanov and M. Safarian. *Markets with transaction costs: Mathematical Theory*. Springer, 2009.
- [23] N. Kaldor. A note on the theory of the forward market. *Review of Economic Studies*, 7(3): 196–201, 1940.
- [24] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 904–950, 1999.
- [25] Peng (Peter) Liu and Ke Tang. No-arbitrage conditions for storable commodities and the modeling of futures term structures. *Journal of Banking & Finance*, 34(7):1675 – 1687, 2010.

-
- [26] H. Pham. Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. *Applied Mathematics and Optimization* 46.1 (2002): 55–78.
- [27] H. Pham. Continuous-time stochastic control and optimization with financial applications. Vol. 1. Berlin: Springer, 2009.
- [28] H. Prevot, B. de Juvigny, F. Lehmann, M. Louvot, and C. Izart. Rapport d'enquête sur les prix de l'électricité. Document, French Ministry of Economy, Finance and Industry, October 2004.
- [29] P.E. Protter. Stochastic Integration and Differential Equations: Version 2.1. Vol. 21. Springer, 2004.
- [30] B. R. Routledge, D. J. Seppi, and C. S. Spatt. Equilibrium forward curves for commodities. *Journal of Finance*, 55(3):1297–1338, 2000.
- [31] J. A. Scheinkman and J. Schechtman. A simple competitive model with production and storage. *Review of Economic Studies*, 50(162):427, 1983.
- [32] E. Schwartz. The stochastic behavior of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3):923–973, 1997.
- [33] J. Viehmann. Risk premiums in the German day-ahead electricity market. *Energy Policy*, 39(1):386 – 394, 2011.
- [34] H. Working. Price relations between july and september wheat futures at Chicago since 1885. *Wheat Studies of the Food Research Institute*, 9(6):187–238, 1933.
- [35] H. Working. The Theory of Price of Storage. *American Economic Review*, 39(6): 1254–1262, 1949.

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