Hotelling Game on Networks

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Since the seminal work of **Hotelling in 1929**, the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets (Spokes, Graitson, Osborne, Salop, Tabuchi, ...).

Hotelling presented a village represented by a line segment where a uniformly distributed continuum of consumers buy a single commodity.

The consumers have to support linear transportation costs when buying the commodity in one of the two firms of the village.

The firms compete in a two-stage location-price game:

1. in the first stage firms choose their location;
2. in the second stage firms choose their prices.

The **goal** of Hotelling’s model is to study the **location** and **price** strategies that maximizes the profit of the competing firms.
Hotelling concluded that firms would agglomerate at the center of the line, an observation referred to as the **Principle of Minimum Differentiation**.

In 1979, D’Aspremont et al. showed that the Principle of Minimum Differentiation is invalid, since there is **no** price equilibrium solution for all possible locations of the firms, in particular when they are not far enough from each other.
The Hotelling model on a line

Brief description
The buyers of a commodity will be supposed uniformly distributed along a line with length $l$.

In the two ends of the line there are two firms $A$ and $B$ selling the same commodity with unitary production costs $c_A$ and $c_B$.

No customer has any preference for either seller except on the ground of price plus transportation cost $t$. 

Figure: Hotelling's linear village
We will assume that each consumer buys a **single unit** of the commodity, in each unit of time and in each unit of length of the line.

Denote A’s **price** by \( p_A \) and B’s **price** by \( p_B \).

The **point of division** \( x = x(p_A, p_B) \in ]0, l[ \) between the regions served by the two entrepreneurs is determined by the condition that at this place it is a matter of indifference whether one buys from A or from B.
**Hotelling model**

Hence, the point of division $x$ is the location of the consumer indifferent to buy from firm A or firm B, if

$$p_A + tx = p_B + t(l - x)$$

Solving for $x$, we obtain

$$x = \frac{p_B - p_A + tl}{2t}.$$

Both firms have a **non-empty market share** if, and only if,

$$0 < x < l.$$
Assuming that both firms $A$ and $B$ have a non-empty demand, the profit of the two firms are given by

$$\pi_A = (p_A - c_A)x = (p_A - c_A) \frac{p_B - p_A + tl}{2t}$$

$$\pi_B = (p_B - c_B)(l - x) = (p_B - c_B) \frac{p_B - p_A + tl}{2t}$$

Hence, the first and second order conditions imply that the optimal prices are

$$p_A = tl + \frac{1}{3}(2c_A + c_B)$$

and

$$p_B = tl + \frac{1}{3}(c_A + 2c_B).$$
Equilibrium Prices and profits

Hence, the prices \((p_A, p_B)\) are a local strategic optimum equilibrium, if the indiferent consumer \(x\) satisfies

\[
0 < x = \frac{p_B - p_A + t l}{2 t} = \frac{l}{2} + \frac{c_B - c_A}{6 t} < l. \tag{1}
\]

Definition

The Hotelling model satisfies the bounded costs (BC) condition, if

\[|c_A - c_B| < 3 t l.\]

Noting that inequality (1) is equivalent to the BC condition, we obtain the following result.

Theorem

Under the BC condition, the pair of prices \((p_A, p_B)\) is a local strategic optimum equilibrium.
The Hotelling town

Extending the Hotelling model from a line to a network
Other models have been developed replacing the line in the Hotelling model by other topologies, as for example in the Salop Model where the line is replaced by the circle.
In this work, we introduce the **Hotelling town model**, that extends the Hotelling model from a **line** (village) to a **network** (town).

Trying to mimic a real town, the roads of the town are the **edges** of the network, the crossroad are the **vertices** with degree higher than two and the ends of no-exit roads are the vertices with degree one.

The firms are spread over the town and the consumers are uniformly distributed along the **roads** (similar size houses).

The roads can have different lengths (**market sizes**) and the firms can have different productions costs (**firm’s heterogeneity**).
We note that our model is different from the usual network games (Bramoulle, Gal and Vega, Goyal, ...), because the consumers are assumed uniformly distributed along the edges of the network and not at the nodes.

A strategy is a price vector that associates to each firm its selling price for the commodity.

As in the original Hotelling model, the expenditure of a consumer that chooses to buy in a firm (shop) consists in the sum of the price practiced by that firm plus the transportation cost that is proportional to the minimal distance between his house (position at the network) and the firm.

Again, the firms compete in a two-stage location-price game:
1. in the first stage firms choose their location;
2. in the second stage firms choose their prices.

The goal is to study the location and price strategies that maximizes the profit of the competing firms.
Introduction

In the price subgame, our main goal is to compute the price strategy that has the following two essential economic properties:

1. **Local strategic optimum**: any small deviation of a price of a firm provokes a decrease in its own profit;
2. **Local market structure**: all firms have a non-empty market.

Property (i) stabilizes the prices because no firm has an incentive to do small changes in their prices.

Property (ii) stabilizes the set of competing firms because every firm has a non-empty market (positive profit). Hence, using a sufficiently high entry cost, no firms enter or leave the market.

We call a price strategy satisfying these two properties a local market optimum price strategy.
Introduction

- We introduce the **weak bounded costs (WBC) condition** on the exogenous parameters of the model that gives a bound in terms of the transportation cost and the minimal road length:
  - on the maximum difference between the production costs; and
  - on the maximum difference between the road length.

- Under the WBC condition, we **prove** that the price subgame has a **local market optimum price strategy**. Furthermore, the local market optimum price strategy is unique.

- We note that, under the WBC condition, the firms can not be located too close, and so D’Aspremont et al. 1979’s objection to the existence of a local market optimum price equilibrium does not apply.
For the local market optimum price strategy, we present an explicit **closed formula** and an explicit **series expansion formula**.

The closed and series expansion formulas **explicitly exhibit** how the local market optimum price strategy of a firm depends:

1. on the production costs;
2. road market sizes; and
3. firms locations.

Hence, the **static analysis** of the relevant economic quantities like profit of firms, consumer surplus and welfare can be done rigorously.
Introduction

Assuming that the firms might not know the entire network:

- We say that a firm has **n-space bounded information** if the firm knows the network structure up to n roads from its location, in terms of production costs, node degrees and road sizes.

- We **prove** that a firm with n-space bounded information is able to compute an **n-approximation price** with the property that the n-approximation price converges to the local market optimum price exponentially fast with n.

- Hence, the **influence** of a firm in the local market optimum price strategy of other firm decreases exponentially fast with the distance between these two firms.
Introduction

A local market optimum price strategy might not be a Nash equilibrium price.

- Hence, we introduce the **strong bounded (SB) condition** that in comparison with the weak bounded condition has the additional feature of its **bound** to depend also on the **maximum node degree** of the network.

- Under the SB condition, we **prove** that the local market optimum price strategy is a **Nash price equilibrium strategy**.
Introduction

- In the **location-subgame**, we consider that a location for the firms is **admissible** if it satisfies the weak bounded condition.

- We **assume** that firms, after choosing an admissible location, will practice the local market optimum price strategy.

- Similarly to the original Hotteling model in the line, we prove that the firms located at the ends of **no-exit** roads have an incentive to deviate from their localization.

- However, we prove that the firms located at **crossroads** do not have an incentive to deviate from their localization. In this sense, firms prefer **maximum differentiation**, in contrast with the original Hotteling model in the line.

- We note that this result has some **empirical evidence**, because in real towns the owners of the shops usually prefer to have them located at crossroads.

- Finally, we extend all the previous results to the case of **incomplete information** on the production costs of the firms.
The Hotelling town model
Price formation
The Hotelling town network consists of a group $E$ of roads (edges of the network) where the consumers live and a group $V$ of firms (nodes of the network);

For simplicity, every firm $F_i$ is located at a position $y_i$ in a neighborhood of a vertex (node) $i \in V$.

Every road has two vertices and in a neighborhood of every vertex is located a single firm;

The consumers can buy one unit of the commodity in any firm of the town;

The firms compete in prices à la Hotelling in the town (network).

Figure: Network example with $y_i = 0$ for all $i \in V$
Hotelling model

Hotelling town network

The firm $F_i$ has a **unit cost** $c_i$ and charges the same unit price $p_i$ to all its customers;

The road (edge) $R_{i,j}$ has **length** $l_{i,j}$ and the consumers are uniformly distributed along the road;

$N_i$ is the set of all neighboring vertices $j$ for which there is a road $R_{i,j}$ connecting the vertices;

The cardinal $k_i$ of $N_i$ is the **degree** of the node $i$.

**Figure:** $l_{1,2} = 2; \ l_{2,3} = 1; \ N_2 = \{1, 3, 8, 9, 10, 13, 14\}$
Let $\tilde{l}_{i,j} = d(y_i, y_j)$ be the distance between the locations of firms $F_i$ and $F_j$.

Let $\epsilon = \max_{i \in V} d(i, y_i)$ be the maximum deviation between the node $i$ and the location of the firm $F_i$.

Hence, for every road $R_{i,j}$, the length of the road $l_{i,j}$ is close to the distance $\tilde{l}_{i,j}$ between the firms $F_i$ and $F_j$,

$$l_{i,j} - 2\epsilon \leq \tilde{l}_{i,j} \leq l_{i,j} + 2\epsilon.$$
Price competition

- A **price strategy** $P$ consists in a vector whose coordinates are the competitive prices $p_i$ of each firm $F_i$;

- A consumer located at a point $x$ of the network who decides to buy at firm $F_i$ **spends**

  $$E(x; i, P) = p_i + t d(x, y_i)$$

the price $p_i$ charged by the firm $F_i$ plus the **transportation cost** that is proportional $t$ to the minimal distance measured in the network between the position $y_i$ of the firm $F_i$ and the position $x$ of the consumer;

- Given a price strategy $P$, the consumer will choose to buy in the firm $F_{V(x, P)}$ that minimizes his expenditure

  $$v(x, P) = \text{argmin}_{i \in V} E(x; i, P).$$
Markets and Profits

- The **market**
  \[ M(i, P) = \{ x : v(x, P) = i \} \]
  of firm \( F_i \) consists of all consumers who minimize their expenditures by opting to buy in firm \( F_i \);
- The market size \( S_i(P) \) of firm \( F_i \) is given by the Lebesgue measure of \( M_i(P) \);
- The profit of the firm \( F_i \) is
  \[ \pi_i(P, C) = (p_i - c_i)S_i(P) \]
  where \( p_i \) is the competitive price of firm \( F_i \) determined by the price strategy \( P \), and \( c_i \) is the production cost of firm \( F_i \) determined by the cost vector \( C \).
Local market structure

The **local firms** of a consumer located at a point $x$ in a road $R_{i,j}$ with vertices $i$ and $j$ are the firms $F_i$ and $F_j$.

A price strategy $P$ determines a **local market structure** if every consumer buys from one of his local firms;
If a price strategy $\mathbf{P}$ determines a local market structure then for every road $R_{i,j}$ there is one consumer located at a point $x_{i,j} \in R_{i,j}$ who is **indifferent** to the local firm from which he is going to buy his commodity, i.e.

$$E(x; i, \mathbf{P}) = E(x; j, \mathbf{P}).$$

Hence,

$$\pi_i(\mathbf{P}, \mathbf{C}) = (2t)^{-1}(p_i - c_i) \left( 2t(2-k_i)\epsilon_i + \sum_{j \in N_i}(p_j - p_i + t\tilde{l}_{i,j}) \right).$$
**Bounded length and costs**

- Let $k_M$ be the maximum node degree of the Hotelling town:
  \[ k_M = \max\{k_i : i \in V\} \]

- Let $c_M$ be the maximum production cost of the Hotelling town and $c_m$ be the minimum production cost of the Hotelling town:
  \[ c_M = \max\{c_i : i \in V\} \quad \text{and} \quad c_m = \min\{c_i : i \in V\} \]
  Furthermore,
  \[ \Delta(c) = c_M - c_m \]

- Let $l_M$ be the maximum road length of the Hotelling town and $l_m$ be the minimum road length of the Hotelling town:
  \[ l_M = \max\{l_e : e \in E\} \quad \text{and} \quad l_m = \min\{l_e : e \in E\} \]
  Furthermore,
  \[ \Delta(l) = l_M - l_m \]

- A Hotelling town satisfies the **weak bounded length and costs (WB)** condition, if
  \[ \Delta(c) + t\Delta(l) < t(l_m - 6\epsilon) \]
The Hotelling town **admissible market size** $L$ is the vector whose coordinates are the **admissible local firm market sizes**

$$L_i = k_i^{-1} \sum_{j \in N_i} \tilde{l}_{i,j}.$$
The Hotelling town neighboring market structure $K$ is the matrix whose coordinates are

1. $k_{i,j} = k_i^{-1}$, if there is a road $R_{i,j}$ between the firms $F_i$ and $F_j$; and
2. $k_{i,j} = 0$, if there is not a road $R_{i,j}$ between the firms $F_i$ and $F_j$.

**Theorem**

If the Hotelling town satisfies the WB condition, then there is unique local optimum price strategy given by

$$P^L = \frac{1}{2} \left( 1 - \frac{1}{2} K \right)^{-1} (C + t(L))$$

$$= \sum_{m=0}^{\infty} 2^{-(m+1)} K^m (C + t(L)).$$

where $1$ is the identity matrix.
Nash equilibrium price strategy

Definition
A Hotelling town satisfies the strong bounded length and costs (SB) condition, if
\[ \Delta(c) + t \Delta(l) \leq \frac{(2tl_m - \Delta(c) - 4t\epsilon)^2}{8tk_M(l_M + \epsilon)} - 3t\epsilon. \]

Theorem
If a Hotelling town satisfies the SB condition then there is a unique Hotelling town Nash equilibrium price strategy \( P^* = P^L \).

Hence, the Nash equilibrium price strategy for the Hotelling town satisfying the SB condition determines a local market structure, i.e. every consumer located at \( x \in R_{i,j} \) spends less by shopping at his local firms \( F_i \) or \( F_j \) than in any other firm in the town and so the consumer at \( x \) will buy either at his local firm \( F_i \) or at his local firm \( F_j \).
Let us explicit the dependence of the profit $\pi(\epsilon_i)$ of the firm $F_i$ with the distance $\epsilon_i$ between its location $y_i$ and the corresponding node $i$.

We say that a firm $F_i$ is **node local stable**, if

$$\pi_i(0) > \pi_i(\epsilon_i)$$

with respect to the local optimum price strategy.

A Hotelling network is **firm node local stable**, if every firm in the network is node local stable.
Hotelling Town model

Strategic optimal location

- Let
  \[ Q_{i,j} = \sum_{m=0}^{\infty} 2^{-(m+1)} k_{i,j}^m. \]

- For every node \( i \), with \( k_i = 2 \),
  - let \( j(i) \) be the node with the property that \( y_i \) is at the road \( R_{i,j(i)} \); and
  - let \( v(i) \in N_i \setminus \{j(i)\} \) be the other node in \( N_i \).

- Define the weighted balance
  \[ U_i = \frac{Q_{i,v(i)}}{k_{v(i)}} - \frac{Q_{i,j(i)}}{k_{j(i)}}. \]
Hotelling Town model

Firm Position Stability

Theorem

- **Firms** $F_i$, with node degree $k_i = 1$, are node local unstable.
- **Firms** $F_i$, with $k_i = 2$, are node local unstable, except for networks with zero weighted balance $U_i = 0$.
- **Firms** $F_i$, with $k_i = 3$ and whose neighboring firms have nodes degree greater or equal to 3, are node local stable.
- **Firms** $F_i$, with $k_i \geq 4$ and whose neighboring firms have nodes degree greater or equal to 2, are node local stable.

Let $k_m = \min\{k_i : i \in V\}$ be the minimum node degree of the Hotelling town. Hence, A Hotelling town network satisfying the WB condition and with $k_m \geq 3$ is firm node local stable.
For every $v \in V$, let $I_v$ (finite, countable or uncountable) be the set of production costs types $I_v$ of firm $F_v$.

Let $d q_v(z_v)$ be the probability of the common believes of the other firms on the production costs $c_{z_v}$ of the firm $F_v$.

The Hotelling town production cost $C$ is the vector $(c_1, \ldots, c_{N_v})$ whose coordinates

$$c_v : I_v \to [c_v^m, c_v^M] \subseteq [c_m, c_M] \subseteq \mathbb{R}_0^+$$

are measurable functions.

The Hotelling town average production cost $E(C)$ is the vector $(E(c_1), \ldots, E(c_{N_v}))$ whose coordinates are the expected production costs

$$E(c_v) = \int_{I_v} c_{z_v}^v \ dq_v(z_v) < \infty.$$
A price strategy $\mathbf{P}$ is a vector $(p_1, \ldots, p_{N_v})$ whose coordinates $p_v : I_v \rightarrow \mathbb{R}_0^+$ are measurable functions.

The average $E(\mathbf{P})$ of the price strategy $\mathbf{P}$ is the vector $(E(p_1), \ldots, E(p_{N_v}))$ whose coordinates are the expected prices

$$E(p_v) = \int_{I_v} p_v z_v \, dq_v(z_v).$$
Local optimum price strategy

Theorem

- If the Hotelling town satisfies the WB condition, then there is unique local optimal equilibrium price strategy given by

\[ P^E = \frac{1}{2} \left( C + KE(P^E) + tL \right) \]  

(2)

where

\[ E(P^E) = \frac{1}{2} \left( 1 - \frac{1}{2} K \right)^{-1} \left( E(C) + tL \right). \]

- The local optimal equilibrium price \( P^E \) determines a local market structure.
- If a Hotelling town satisfies the SB condition, then there is a unique Hotelling town Bayesian-Nash equilibrium price strategy \( P^* = P^E \).
Thank you!
Hotelling Town: uncertainty on production costs

- Let the vector
  \[ Z_{N_i} = (z_{i,1}, z_{i,2}, \ldots, z_{i,k_i}) \]
  be the type of the neighbours of a firm \( F_i \) of degree \( k_i \), and let
  \[ I_{N_i} = I_{i,1} \times I_{i,2} \times \ldots \times I_{i,k_i} \]
  be the set of all vectors \( Z_{N_i} \).
- Let
  \[ dq_{N_i}(Z_{N_i}) = dq_{i,1}(z_{i,1}) dq_{i,2}(z_{i,2}) \ldots dq_{i,k_i}(z_{i,k_i}) \]
  be the probability of the belief of the firm \( F_i \) on the production costs of its neighbours to be
  \[ C_{N_i}^{Z_{N_i}} = (c_{i,1}^{z_{i,1}}, c_{i,2}^{z_{i,2}}, \ldots, c_{i,k_i}^{z_{i,k_i}}). \]
Local market structure

- For each road $R_{i,j}$, the indifferent consumer

$$x_{i,j} : I_i \times I_j \to (0, l_{i,j})$$

is given by

$$x_{i,j}^{z_i,z_j} = \frac{p_{j}^{z_j} - p_{i}^{z_i} + t l_{i,j}}{2 t}. \quad (3)$$

- Under a local market structure, the ex-post profit

$$\pi_{i}^{EP} : I_i \times I_{N_i} \to \mathbb{R}_0^+$$

of firm $F_i$ is given by

$$\pi_{i}^{EP}(z_i, Z_{N_i}) = (p_{i}^{z_i} - c_{i}^{z_i}) \sum_{j \in N_i} x_{i,j}^{z_i,z_j}.$$
Local market structure

Under a local market structure, the ex-ante profit

\[ \pi^\text{EA}_i : I_i \to \mathbb{R}_0^+ \]

of firm \( F_i \) is given by

\[ \pi^\text{EA}_i(z_i) = \int_{I_{N_i}} \pi^\text{EP}_i(z_i, Z_{N_i}) \, d\nu_{N_i}(Z_{N_i}) \]

\[ = (p_i^{z_i} - c_i^{z_i}) \sum_{j \in N_i} \frac{E(p_j) - p_i^{z_i} + t l_{i,j}}{2t} \]