Gas Storage Hedging

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1 Introduction

Gas storage valuation has been an intense subject of research during the recent years. This problem is related to optimal control problems [17, 15] and more precisely to the class of optimal switching problem. On the energy market, the gas storage management can be seen as a so called Swing option [12] with some operational constraints : each day the manager of the gas storage has to decide either to inject gas in the storage, which he bought on the gas market, or to withdraw gas from the storage to sell it on the market, or to do nothing. Moreover it has to deal with some operational constraints :

- compressor used to inject or withdraw gas may break down,
- in order to preserve the storage cavity used due to thermomechanical constraints for example in a salt cavity, complete withdrawal can not be achieved at full speed,
- the formation of hydrates has to be limited for safety reasons.

Respecting these constraints, the manager will try to maximize its earnings on average having to deal with the stochasticity of the gas prices.
Historically, gas storage valuation was coarsely valuated as a strip of call spread options [9] totally ignoring operational constraints and some properties of the storage. Because of the fact that injection capacity decreases as the storage is filled and the withdrawal capacity decreases as the the pressure in the cavity decreases, the problem is fully non linear and the classical dynamic programming method [3] is generally used as a solver. In fact these physical constraints assure convexity of the problem and permit also to use the stochastic dual dynamic programming method [16, 19] : some commercial software as QEM use this approach [20]. During the

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late nineties, a lot of research has been devoted to the search of efficient numerical procedures to solve the Swing option problem. First trinomial trees were used to solve these problems [12], then Longstaff Schwartz method [13] and Partial Differential Equation methods [21] were applied to solve these problems. As for the more complex case of gas storage valuation, during recent years Monte Carlo methods [2, 14] and PDE methods [6] have been used to accurately valuate the assets. Practitioners now classically use Monte Carlo to price Swing options [11] or gas storage but in order to calculate the hedge at the valuation date they use classical finite difference methods. Besides practitioners are interested in evaluating the expected effectiveness of their hedge. Most of the time classical mean reverting one or two factor models are used to describe gas price models [18, 7]. But even within the Black Scholes framework some source of incompleteness occurs:

- Hedging is permitted only once a day. As explained for example in [4], future and spot prices are settled only once a day even if trading is possible continuously. The hedging periodicity can not be shorter than one day and it is well know that the hedging error in Black Scholes framework converges to zero at a rate proportional to the square root of hedging frequency [22, 10].
- Daily futures contracts are not available for the following days. Day ahead product can be seen as spot and depending on the market,
  - either only monthly future products are available for the 72 following months for example at Henry Hub,
  - or monthly future products are available for the 10-12 following months, 11-12 quarter products and 6 seasons for example at Intercontinental Exchange. A season corresponds either to summer ranging from April to September, or to winter ranging from October to March.
  - or the three following months, the two following quarters and three following quarters for example at Powernext.

  Notice that even if week ahead products are not quoted on the market for hedging till the end of the month, it is possible to find such products traded over the counter.

For these two reasons even in the "perfect gaussian world" hedge can not be perfect and practitioners are very interested in studying its effectiveness. In order to simulate the hedging strategy, practitioners generate some price scenarios. They try to use classical finite difference methods to compute the hedging strategy on each scenario each day of the studied period for all future products available. Even with powerful clusters, this task cannot be achieved in reasonable time.
In this article we first recall how to use tangent processes to calculate delta of American options to avoid the use of finite difference and show that the hedge calculated is efficient. In a second part we explain how to use this tangent process to accurately calculate conditional delta for gas storage. We give algorithms to calculate efficiently this conditional delta and explain how to deal with the dynamic of the real future products available on the market. We explain how to use this conditional delta in simulation and give some numerical results showing its efficiency for salt

cavities and depleted gas fields even in the case where only a few future products are available. To our knowledge it is the first time that numerical results are given for the hedge of Swing style options with conditional delta. We compare our approach with the classical finite difference and tangent process methods for hedging the considered storage assets on various gas price scenarios and show that a calculation that would take more than one year by other methods can be achieved in a few minutes.

2 Recall on American and Bermudan Options and Delta Hedging

In this section we recall some results on delta hedging of American and Bermudan options and we give the algorithms used to efficiently hedge theses options.

2.1 Formulas

In this section most of the theoretical background can be found in the articles listed in [5].
In the sequel we suppose that conditional expectation are calculated by the Longstaff-Schwartz method [13] adapted as explained in [5]. The algorithms given here are associated to this method and so, are based on a Monte Carlo estimation of the option valuation and its hedging. Using trees methods for example would lead to another representation of the conditional delta (the one presented here is heavily using the estimation of optimal exercise of American type options). For some other representations of the delta see [5] and references inside.
All over this section, we shall consider a one-dimensional Brownian motion W on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the natural (completed and right-continuous) filtration $\mathbb{F} = (\mathscr{F}_t)_{t \le T}$ generated by W up to some fixed time horizon $T > 0$. The dynamic of the stock is given by

$$S_t = S_0 \int_0^t \sigma(s) S_s dW_s \quad t \le T, \tag{1}$$

where σ is a Lipschitz continuous function defined on $\mathbb{R}$ taking values in $\mathbb{R}$. In the case of American options with pay off g, maturity T, the risk free rate being taken equal to 0, it is well known that the price of an American option at time t P_t is given by

$$P_t = \text{esssup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}\left[g(S_\tau) \mid \mathscr{F}_t\right] \quad \text{for } t \le T \;\; \mathbb{P}-\text{a.s.}\,, \tag{2}$$

where $\mathscr{T}_{[t,T]}$ denotes the set of $\mathbb{F}-$stopping times with values in $[t,T]$. As recalled in [5] with the tangent process Y_t solution of this equation

$$\begin{cases} dY_t = & \sigma(t) Y_t dW_t, \\ Y_0 = & 1, \end{cases} \tag{3}$$

the delta can estimated using this representation if g and σ belong to the set of differentiable functions with continuous and uniformly bounded derivatives C_b^1:

$$\Delta_t = \mathbb{E}\left[g'(S_{\tau_t}) Y_{\tau_t} \mid \mathscr{F}_t\right] (Y_t)^{-1} \;, \; t \leq T \;. \tag{4}$$

where τ_t is the optimal exercise time after t such that

$$\mathbb{E}\left[g(S_{\tau_t}) \mid \mathscr{F}_t\right] = \text{esssup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}\left[g(S_\tau) \mid \mathscr{F}_t\right].$$

This results remains true if g can be uniformly approximated by a sequence of C_b^1 functions.
The finite difference approach consists in estimating the price process for different initial conditions. More precisely, let S^ε be defined for $\varepsilon \in \mathbb{R}$ as

$$S_t^\varepsilon = (S_0 + \varepsilon) \int_0^t \sigma(s) S_s^\varepsilon dW_s \quad t \leq T, \tag{5}$$

and P^ε be defined as the solution of

$$P_t^\varepsilon = \text{esssup}_{\tau \in \mathscr{T}_{[t,T]}} \mathbb{E}\left[g(S_\tau^\varepsilon) \mid \mathscr{F}_t\right] \quad \text{for } t \leq T \;\; \mathbb{P}-\text{a.s.}\,. \tag{6}$$

Then, following the standard approach for European options, one can approximate Δ_0 by $(P_0^\varepsilon - P_0)/\varepsilon$ or $(P_0^\varepsilon - P_0^{-\varepsilon})/2\varepsilon$ where $\varepsilon > 0$ is a small scalar. A large literature is available on this approach for European type options, see e.g. [8] and the references therein.

2.2 Classical Longstaff-Schwartz and Conditional Delta

In what follows, we approximate the value of the American option by estimating the optimal exercise time of the corresponding Bermudan option with grid with time step π: $\{t_i = i\pi : i = 0, \ldots, \kappa\}$. We denote by $\hat{\mathbb{E}}[\cdot \mid \mathscr{F}_{t_i}]$ an approximation of the true conditional expectation operator $\mathbb{E}[\cdot \mid \mathscr{F}_{t_i}]$ using a regression based approach. $\hat{\tau}$ will represent the estimation of the optimal exercise time associated to the Bermudan option. We denote by M the number of Monte Carlo simulations. $S_i^j = S_{t_i}^j, Y_i^j = Y_{t_i}^j i = 0, \ldots, \kappa$, $j = 1, \ldots, M$ is the j-th simulation of the asset at time step i corresponding to date $t_i = i\pi$. Algorithm 1 estimates the option value, the delta at initial date 0 and calculates conditional delta during optimization. This is a classical dynamic programming method [3] : at each time step, we compare the gain obtained by immediate exercise with the expected gain if the exercise is postponed. With exercising, cash flows on trajectories are updated. This methodology is equivalent to keep in memory the optimal exercise time on each trajectory. At time step zero, cash flows generated on each trajectory at each optimal exercise times are averaged to give the option value. Besides, during this backward recursion, conditional delta Δ are stored using a regression approach to approximate (4). In the sequel we call conditional delta at date $i\pi$ an approximated function giving the hedge at date $i\pi$ depending on the asset price if the option has not been exercised at this date. The hedge at initial date is simply obtained by average.

Algorithm 1 Price an American option, compute delta and conditional deltas

Require: Option and asset parameters
Ensure: Calculate the option price and conditional deltas at each time step
  $CF^j = g(S^j_\kappa)$ for $j = 1$ to M // final cash flow
  $D^j = Y^j_\kappa g'(S^j_\kappa)$ for $j = 1$ to M // final delta
  for $i = \kappa - 1$ to 0 **do**
    Calculate and store cash flow conditional expectation $Esp_i(s) = \hat{\mathbb{E}}[CF|\, S_{i\pi} = s]$
    Calculate and store conditional deltas $\Delta_i(s) = \hat{\mathbb{E}}[D|\, S_{i\pi} = s]/Y_i$
    for $j = 1$ to M **do**
      if $Esp^j_i < g(S^j_i)$ **then**
        $CF^j = g(S^j_i)$
        $D^j = Y^j_i g'(S^j_i)$
      end if
    end for
  end for
  Estimation of the option value $\frac{\sum_j^M CF^j}{M}$ and delta $\frac{\sum_j^M D^j}{M}$.

Algorithm 2 simulates the optimal exercise of the American option and its hedging for some Monte Carlo scenarios S^j_i, for $i = 0,\dots,\kappa$, $j = 1,\dots,M$.
In order to validate this approach, we use the classical Black-Scholes model for an asset S_t following classical SDE

$$dS_t/S_t = \mu dt + \sigma dW_t, \tag{7}$$

with trend $\mu = 0.1$, volatility $\sigma = 0.2$, risk free rate $r = 0.05$. We study the hedging effectiveness of at the money put option with strike 1 with maturity one year. The value of the option is 0.06104. We can check on numerical results in Table 1 that as the frequency of the hedge inscreases, the expected gain obtained with hedging converges to the value of the option and that the standard deviation of the gain goes to zero. The number of simulations used for results in Table 1 in optimization and simulation is taken equal to one million and the number of time steps used to valuate the American option is taken equal to 720 so that results are converged.

Remark 1. Of course if $\mu = r$, we would always have an average hedging portfolio equal to the option value.

Algorithm 2 Simulate and hedge an American option

Require: Continuation values and conditional delta computed in optimization part, see Algorithm 1

Ensure: Simulate the option exercise, portfolio with and without hedging for one scenario k

```
v = 0. // Portfolio without hedge initialization
v_h = 0. // Portfolio with hedge initialization
bcash(j) = true for j = 1 to M // Continue time step exploration till exercise
for i = 0 to κ−1 do
    Get back continuation function at time i stored in Algorithm 1 Esp_i(.)
    Get back conditional delta at time i stored in Algorithm 1 Δ_i(.)
    for j = 1 to M do
        if bcash(j) = true then
            if g(S_i^j) > Esp_i(S_i^j) then
                v = g(S_i^j)
                v_h = v_h + g(S_i^j)
                bcash(j) = false
            else
                v_h −= Δ_i(S_i^j)(S_{i+1}^j − S_i^j)
            end if
        end if
    end for
end for
for j = 1 to M do
    if bcash(j) = true then
        v_h = v_h + g(S_κ^j)
        v = g(S_κ^j)
    end if
end for
```

Number of hedges	1	5	10	20	40	80	160
Average cash flow with hedge	0.04822	0.06176	0.06138	0.06117	0.06111	0.06108	0.06106
Standard deviation	0.06605	0.02997	0.02369	0.01933	0.01656	0.01490	0.01397

Table 1 Efficiency of the delta hedging with the conditional tangent process for the Black-Scholes model

3 Gas Storage Valuation and Hedging Methodology

We first give in this section the price model used for valuation and hedging. Similarly to the previous section we recall the mathematical formulation of the gas storage valuation problem in a continuous time framework. In a second part we give the algorithm used to valuate a gas storage asset by dynamic programming. We explain how to calculate Bellman values and conditional deltas associated to the problem. At last we give the algorithm used to hedge the asset taking into account the availability of the futures products on the market.

3.1 Price Model

All over this section, we shall consider a n-dimensional Brownian motion $\{z^1,\dots,z^n\}$ with correlation matrix ρ on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ endowed with the natural (completed and right-continuous) filtration $\mathbb{F}=(\mathscr{F}_t)_{t\le T}$ generated by $\{z^1,\dots,z^n\}$ up to some fixed time horizon $T>0$.

3.1.1 Future Price Model

We suppose that the daily price of gas S_t follows under the risk neutral measure a n-dimensional Ornstein-Uhlenbeck process [7]. The following SDE describes our uncertainty model for the forward curve $F(t,T)$ giving the prices of a unitary amount of gas at day t for delivery at date T:

$$\frac{dF(t,T)}{F(t,T)}=\sum_{i=1}^{n}\sigma_i(t)e^{-a_i(T-t)}dz_t^i, \tag{8}$$

with σ_i some volatility parameters and a_i mean reverting parameters.

Remark 2. Most of the time a two factors model is used. In this model, the first Brownian motion describes swift changes in the future curve, the second one describes structural changes in the gas market and deals with long term changes in the curve. The mean reverting parameter is generally taken equal to zero for the second term.

With the following notations:

$$V(t_1,t_2)=\int_0^{t_1}\left\{\sum_{i=1}^{n}\sigma_i(u)^2e^{-2a_i(t_2-u)}+2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\rho_{i,j}\sigma_i(u)e^{-a_i(t_2-u)}\sigma_j(u)e^{-a_j(t_2-u)}\right\}du,$$
$$W_t^i=\int_0^t\sigma_i(u)e^{-a_i(t-u)}dz_u^i,\ \text{for } i=1,\dots,n$$

the integration of equation (8) gives :

$$F(t,T)=F(t_0,T)e^{-\frac{1}{2}V(t,T)+\sum_{i=1}^{n}e^{-a_i(T-t)}W_t^i}. \tag{9}$$

With this modelization, the spot price is defined as the limit of the future price :

$$S_t=\lim_{T\downarrow t}F(t,T). \tag{10}$$

In the sequel we note $X_t=(W_t^1,\dots,W_t^n)$ the stochastic state vector.

3.1.2 Tangent Process

Similarly to the Black-Scholes model we introduce the " forward tangent" process noted Y_t^T satisfying :

$$Y_t^T = e^{-\frac{1}{2}V(t,T)+\sum_{i=1}^n e^{-a_i(T-t)}W_t^i}. \tag{11}$$

With this new notation, the conditional delta at date t dealing with delivery at date T for a classical European option with pay off g belonging to C_b^1 can be written:

$$\Delta_t^T = \mathbb{E}\left[g'(S_T)Y_T^T \mid \mathscr{F}_t\right]/Y_t^T \ \ , t \leq T \ . \tag{12}$$

3.2 Gas Storage Modelization

We denote by C_t the gas volume level in the gas storage at date t. We suppose that the asset management is such that C_t satisfies $C_{min} \leq C_t \leq C_{max}$. At each volume level C_t a set of possible commands is associated. Noting a_{in} the injection rate per time unit (a day typically), and a_{out} the withdrawal rate, this command to be executed during a time step π is chosen inside the interval $[\max(-a_{out}\pi, C_{min} - C_t), \min(a_{in}\pi, C_{max} - C_t)]$.

In the sequel we suppose that a bang-bang strategy is used. This strategy is a good approximation of the optimal strategy [2, 1]. In [6] some numerical examples are given to estimate the error associated with this supposition on real assets. This strategy when not dealing with global constraints C_{min} and C_{max} supposes that at each time step, the optimal command per time unit is to be chosen so that gains per time unit are between the following ones :

$$\begin{cases} \text{Injection} & a_{in} & \text{for a gain of } \phi_{-1}(S_t) = -S_t a_{in} - K_{in} \quad (\text{regime } -1) \\ \text{Do nothing} & & \text{for a gain of } \phi_0(S_t) \ \ = -K_s \quad (\text{regime } 0) \\ \text{Withdraw} & a_{out} & \text{for a gain of } \phi_1(S_t) \ \ = S_t a_{out} - K_{out} \quad (\text{regime } 1) \end{cases} \tag{13}$$

where K_{in}, K_s and K_{out} are some costs that we will suppose null to alleviate notations in the storage modelization.

Remark 3. Notice that the injection rate a_{in} and the withdrawal rate a_{out} depends on the volume level C_t. To simplify the notations we drop the dependence on C_t.

At date t the regime u_t can take three values

- 1 in withdrawal mode,
- 0 in storage mode,
- -1 in injection mode.

We note u a given strategy to manage the asset: this strategy consists in a set of regimes and some associated stopping time where regimes change. We note u_t the regime number at date t belonging to $\{-1, 0, 1\}$. For this given strategy u, knowing that at date t the factors of the price model are given by $x = (w^1, \ldots, w^n)$, that the volume level is c , neglecting switching cost, the expected profit associated to the asset management between dates t and T is :

$$J(t,x,c;u) = \mathbb{E}\left[\int_t^T \phi_{u_r}(S_r)dr + J(T,X_T,C_T;u_T)|X_t = x, C_t = c\right] \quad (14)$$

In the sequel we will suppose that the final value of the asset is 0. The asset operator will try to find a strategy in the set $\mathscr{U}_t$ of the admissible non anticipative strategies in order to maximize its gains and solve the problem

$$J^*(t,x,c) = \sup_{u \in \mathscr{U}_t} J(t,x,c;u). \quad (15)$$

3.2.1 Dynamic Programming and Daily Hedging for Gas Storage

We suppose in the sequel that dates where the regime switches are allowed are discrete $t_i = i\pi$, for $i = 0,\dots,\kappa-1$ and $\kappa = T/\pi$. According to the Bellman principle, at a date t_i, for some given random factors $x = (w^1,\dots w^n)$, the value of the asset at a given time step $t_i = i\pi$ with a volume level c follows :

$$J^*(t_i,x,c) = \sup_{k\in\{-1,0,1\}} \left\{\phi_k(S_{t_i})\pi + \mathbb{E}\left[J^*(t_{i+1},X_{t_{i+1}},\tilde{c}_k) \mid X_{t_i} = x, C_{t_i} = c\right]\right\} \quad (16)$$

where

$$\begin{aligned}
\tilde{c}_{-1} &= \min(c + a_{in}\pi, C_{max}),\\
\tilde{c}_0 &= c,\\
\tilde{c}_1 &= \max(c - a_{out}\pi, C_{min}).
\end{aligned}$$

We define $V^*(t,x,c)$ the optimal volume exercised at date t, with $X_t = x$ and $C_t = c$, and taking three different possible values: $-a_{in}\pi$ if injection is optimal, $a_{out}\pi$ if withdrawal is optimal, 0 if idle is optimal (when ignoring the constraints on the stock level). We note $C_m^{*,i}(x,c)$ the optimal volume level at date t_m starting at level c at date t_i following X_t trajectory with $X_{t_i} = x$. The optimal volume level $C_m^{*,i}(x,c)$ is $\mathscr{F}_{t_m}$-mesurable and follows

$$\begin{aligned}
C_i^{*,i}(x,c) &= c,\\
C_m^{*,i}(x,c) &= c - \sum_{k=i}^{m-1} V^*(t_k,X_{t_k},C_k^{*,i}(x,c)) \text{ for } 0 \le i \le m \le \kappa.
\end{aligned} \quad (17)$$

Thus $\sum_{k=0}^{\kappa-1} V^*(t_k,X_{t_k},C_k^{*,0}(x,c))$ corresponds to the sum of the optimal volumes exercised following the optimal strategy starting from a volume c at time step 0 where $X_{t_0} = x$.
Noticing that the optimal volume exercised corresponds to the derivative of the gain function $\pi\phi$ in the optimal regime in equation (16), in view of equation (4), we

introduce the $\mathscr{F}_{t_m}$-adapted random variable:

$$D(t_i, t_m, x, c) = V^*(t_m, X_{t_m}, C_m^{*,i}(x,c)) Y_{t_m}^{t_m}. \tag{18}$$

Conditional delta at date t_i for delivery at date t_m , $m = i+1, \dots \kappa - 1$ is easily calculated by equation (12)

$$\Delta(t_i, t_m, x, c) = \mathbb{E}[D(t_i, t_m, X_{t_i}, c) \mid X_{t_i} = x] / Y_{t_i}^{t_m}. \tag{19}$$

In order to solve equation (16), the classical dynamic backward programming method using Longstaff-Schwartz methodology [13] can be used as in the case of American options. The main difficulty comes from the fact that we do not know what the volume is at a future date $\tilde{t}$. This volume level depends on the strategy applied between dates t and $\tilde{t}$. So, in the optimization part, we have to store for each volume level the Bellman values and the hedging strategies associated as shown by equation (16).
The volume is discretized on a grid

$$c_l = C_{min} + l\delta, \quad l = 0, \dots, L = (C_{max} - C_{min})/\delta$$

where δ is the mesh size. Similarly to the case of American option, a Monte Carlo method is used to get some gas prices simulations S_i^j for $j = 1, \dots, M$ at dates t_i. Conditional expectation is estimated using regression as in [5] and cash flows are estimated as in Subsection 2.2.
We note $\hat{V}^*$ and $\hat{C}^*$ the estimation of the optimal volume V^* and optimal volume levels C^* obtained by the Longstaff-Schwartz method and note $\hat{D}$ the function storing the optimal volume $\hat{V}^*$ multiplied by tangent process along trajectories :

$$\hat{D}(t_i, t_m, X_{t_i}^j, c_l) = \hat{V}^*(t_m, X_{t_m}^j, \hat{C}_m^{*,i}(X_{t_i}^j, c_l)) Y_{t_m}^{t_m, j}. \tag{20}$$

The $\hat{D}$ values at date t_i can be calculated by the following backward recursion knowing $\hat{D}(t_{i+1}, t_m, X_{t_{i+1}}^j, c_l)$ for $m = i+1, \dots \kappa - 1$, $l = 1, \dots L$, $j = 1, \dots M$ and the optimal volume to exercise $\hat{V}^*(t_i, X_{t_i}^j, c_l)$:

$$\begin{aligned} \hat{D}(t_i, t_i, X_{t_i}^j, c_l) &= \hat{V}^*(t_i, X_{t_i}^j, c_l) Y_{t_i}^{t_i, j} \\ \hat{D}(t_i, t_m, X_{t_i}^j, c_l) &= \hat{D}(t_{i+1}, t_m, X_{t_{i+1}}^j, c_l - \hat{V}^*(t_i, X_{t_i}^j, c_l)), m = i+1, \dots \kappa - 1 \end{aligned} \tag{21}$$

Remark 4. Equation (21) is simply obtained by using equation (20) and the fact that $\hat{C}_m^{*,i}(X_{t_i}^j, c_l) = \hat{C}_m^{*,i+1}(X_{t_{i+1}}^j, c_l - \hat{V}^*(t_i, X_{t_i}^j, c_l))$ for $\kappa > m > i$.

Remark 5. The $\hat{D}$ value is an approximation in equation (21) :

- $\hat{D}$ is the result of the approximated Longstaff-Schwartz procedure,
- $\hat{D}$ is only available at some discretized stocks points and an interpolation in the delta values is needed when optimal volume reached does not correspond to discretization points of the storage capacity.

The Longstaff-Schwartz estimator of the conditional delta is then evaluated for $m > i$ by

$$\hat{\Delta}(t_i, t_m, x, c_l) = \hat{\mathbb{E}}[\hat{D}(t_i, t_m, X_{t_i}, c_l) \mid X_{t_i} = x]/Y_{t_i}^{t_m}. \tag{22}$$

Algorithm 3 gives the entire procedure to compute the gas storage value and the deltas. It is a simplified one not dealing with special cases and interpolation needed between volume levels.

3.2.2 Cash Flow Simulation and Delta Hedging

As for American options, an algorithm can be derived to calculate the cash flow generated by the gas storage management and by the delta hedging. Because the model is a daily model, hedging could be theoretically done with daily products if daily products were available on the future market for all maturities. As we will see later on numerical results the algorithm can be used with the real products available on the gas market.
Two options are possible :

- Calculate as proposed in algorithm 3 the daily deltas. Then do as in Algorithm 4 and approximate the Δ at date t for a future product p with delivery period $\mathscr{P}_p$ as the sum of all the daily delta ponderated by the future value at the date t :

$$\Delta_p(t) = \frac{\sum_{t_i \in \mathscr{P}_p} \hat{\Delta}(t, t_i) F(t, t_i)}{\sum_{t_i \in \mathscr{P}_p} F(t, t_i)}. \tag{23}$$

- Instead of aggregating the daily deltas in simulation it is possible to modify Algorithm 3 directly to store the hedge for the products available during optimization. Because the conditional delta in Algorithm 3 at date t are calculated for each date $\tilde{t} > t$, the data storage can become cumbersome if maturity of the asset is longer than a few months. It is possible to reduce the number of conditional expectations to calculate and data storage if the future products satisfy some rules described below.
  We note $\mathscr{Q}_t$ the set of future products available at date t, and for all $p \in \mathscr{Q}_t$, $\mathscr{P}_p$ the delivery period associated to product p, η_p the beginning of the delivery period. Supposing that $\forall t > 0, \forall p \in \mathscr{Q}_t, \forall \tilde{t} > t$ there exist $\mathscr{Q}^p \subset \mathscr{Q}_{\tilde{t}}$ such that $\mathscr{P}_p = \cup_{\tilde{p} \in \mathscr{Q}^p} \mathscr{P}_{\tilde{p}}$ then it is possible to aggregate an approximated conditional delta at date t per product with an ad hoc rule so that a dynamic programming approach is still usable :

$$\begin{aligned}
\tilde{D}(t_i, p, x, c) &= \hat{\mathbb{E}}[\sum_{t_m \in \mathscr{P}_p}^{\kappa} \hat{V}^*(t_m, X_{t_m}, \hat{C}_m^{*,i}(X_{t_i}, c)) Y_{t_m}^{t_m} F(0, t_m) \mid X_{t_i} = x] \\
\tilde{\Delta}(t_i, p, x, c) &= \tilde{D}(t_i, p, x, c) / (Y_{t_i}^{\eta_p} \sum_{t_m \in \mathscr{P}_p} F(0, t_m))
\end{aligned} \tag{24}$$

where $\tilde{\Delta}(t_i, p, x, c)$ represents the power to invest at date t_i for product p for a gas volume level c and a stochastic state vector x. Noticing that

$$\tilde{D}(t_i, p, x, c) = \hat{\mathbb{E}}[1_{t_{i+1} \in \mathscr{P}_p} \hat{V}^*(t_{i+1}, X_{t_{i+1}}, \hat{C}^{*,i}_{i+1}(X_{t_i}, c)) Y^{t_{i+1}}_{t_{i+1}} F(0, t_{i+1}) \mid X_{t_i} = x]$$
$$+ \sum_{\tilde{p} \in \mathscr{Q}^p} \hat{\mathbb{E}}[\sum_{t_m \in \mathscr{P}_{\tilde{p}}, m > i+1} \hat{\mathbb{E}}[\hat{V}^*(t_m, X_{t_m}, \hat{C}^{*,i}_m(X_{t_i}, c)) Y^{t_m}_{t_m} F(0, t_m) \mid X_{t_{i+1}}] \mid X_{t_i} = x],$$

we get

$$\tilde{D}(t_i, p, x, c) = \hat{\mathbb{E}}[1_{t_{i+1} \in \mathscr{P}_p} \hat{V}^*(t_{i+1}, X_{t_{i+1}}, \hat{C}^{*,i}_{i+1}(X_{t_i}, c)) Y^{t_{i+1}}_{t_{i+1}} F(0, t_{i+1}) \mid X_{t_i} = x]$$
$$+ \sum_{\tilde{p} \in \mathscr{Q}^p} \hat{\mathbb{E}}[\tilde{\Delta}(t_{i+1}, \tilde{p}, X_{t_{i+1}}, c - \hat{V}^*(t_i, X_{t_i}, c)) \mid X_{t_i} = x].$$

The function $\tilde{D}$ can be evaluated by backward recursion.

Remark 6. The direct use of equation (23) is not possible within a dynamic programming framework.

In Algorithm 4, we simulate an hedged portfolio with M new simulations prices S^j_i $j = 1, \ldots M$ generated by Monte Carlo under the real world probability. $F^j(t, T)$ is the forward curve seen at date t for delivery at date $T \geq t$ and simulation $j \in [1, M]$. The given algorithm does not suppose the previous aggregation property. It can be used after Algorithm 3 and corresponds to the first option developped in this section.

4 Numerical Results

In a first part we give the parameters used for gas modeling, the parameters used to describe two kinds of gas storage and the hedging products. In a second part, we compare our method, which will be called “conditional tangent process method” to the finite difference method and the tangent process method.

4.1 Market Representation

All numerical results are given with a two factor model. Parameters for this model are given in Table 2 and the initial forward curve is given in Fig. 1 starting the first of July 2010. The annual interest rate is taken equal to 6%.

We assumed daily hedging. We assumed the following:

- Daily products are available for delivery from tomorrow to the end of the running week.
- Weekly products are available for delivery from next week to the end of the month. For simplification purpose, we avoid overlapping products by truncating the last available week product at the last day of the month.

long term volatility	29 % / $\sqrt{year}$
long term mean reverting	0 / *year*
short term volatility	94 % / $\sqrt{year}$
short term mean reverting	7.4 / *year*
correlation	-0.13

Table 2 Gas model parameters

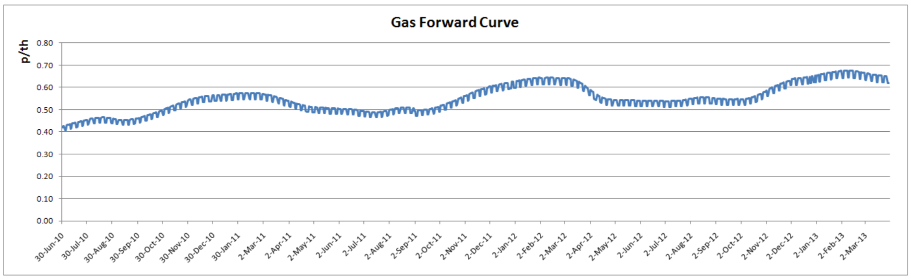


Fig. 1 Initial forward curve

- Monthly products are available from next month and up to the next quarter.
- Quarter products are available for delivery starting next quarter and until next year.
- Finally, next year Y+1, Y+2, and Y+3 are available.

The inclusion property leading to equation (24) is respected, so all calculations with conditional deltas can be carried out on a small laptop.

4.2 Gas Storage Description

Two sets of parameters are used to describe two typical kinds of gas storage: Salt cavities and gas storage tanks will be represented by a "fast storage" set of parameters, whereas depleted gas fields will be described with the "seasonal storage" set of parameters (see Table 3). Prices are given in pence per therm [1] and the different capacities in therms. Fast storages and seasonal storages differ greatly in their injection/withdrawal rate. It requires around two weeks to totally fill and empty a fast storage, against several months for the seasonal storage. Hence, seasonal storages's

[1] The therm (symbol thm) is a non-SI unit of heat energy equal to 100,000 British thermal units (BTU). It is approximately the energy equivalent of burning 100 cubic feet (often referred to as 1 Ccf) of natural gas.

	Fast storage	Seasonal storage
Working gas capacity C_{max} (th)	36,600,000	32,637,363
withdrawal rate a_{out} (th/day)	4,500,000	400,000
injection rate a_{in} (th/day)	6,000,000	140,659
withdrawal cost K_{out} (p/th/day)	0.35	0.31
injection cost K_{in} (p/th/day)	0.35	0.72

Table 3 Parameters of the fast and seasonal gas storage assets, C_{min} and K_s being equal to 0.

asset manager will basically arbitrage the seasonal price differences (store in summer, withdraw in winter) (Fig.2), whereas fast storages's will rather take benefit of the volatility and the weekly seasonality (Fig.3). The fast gas storage will be optimized and simulated during one year whereas the seasonal storage will be optimized and simulated during three years.

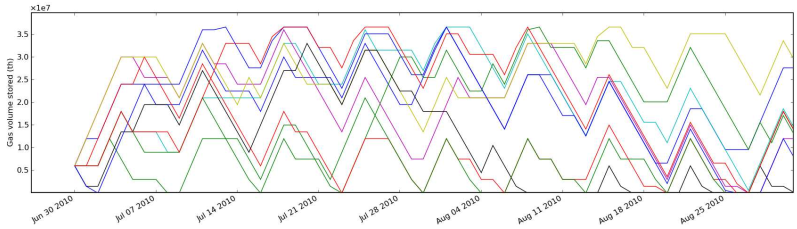


Fig. 2 10 simulations of the optimal fast gas storage levels (two months)

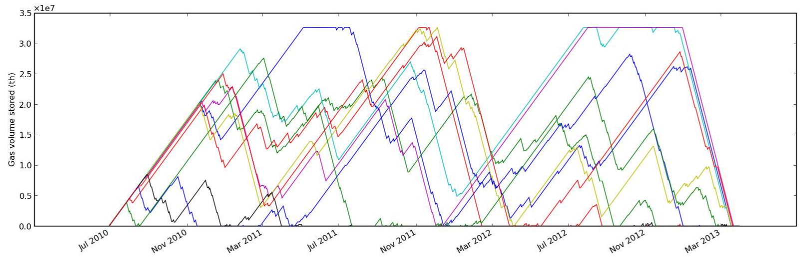


Fig. 3 10 simulations of the optimal seasonal gas storage levels (3 years)

4.3 Comparison between Finite Difference and Tangent Process

All calculations except when specified are achieved with 70000 simulations during the optimization parts with 70 (10×7) local basis functions for the uncertainty $x = (w^1, w^2)$ discretization. The basis functions used are detailled in [5]. As for the gas levels used for valuation, the fast gas storage is discretized with 24 steps, and the seasonal one with 80 steps. We compare the three methods supposing that the forward curve deformation is given by equation (8). During this forward curve deformation the prices of the future products are obtained by averaging the curve on the delivery period.

- In conditional tangent process method, all commands and hedges are calculated during the optimization part,
- As for the tangent process, at each time step, a valuation is achieved calculating the hedge applied at this date using Algorithm 3 not storing conditional hedges.
- As for the finite difference method, at each hedging date t a first calculation is achieved with the current forward curve $F(t,.)$ at this date giving a first valuation V. Then for each available future product with delivery period p, the future curve is shifted by

$$\begin{aligned} F^{\varepsilon}(t,T) &= F(t,T)(1+\varepsilon) \text{ for all } T \in p, \\ &= F(t,T) \text{ if } T \text{ not in } p, \end{aligned}$$

  leading to a second valuation V^{ε}. The product's sensibility is thus given by $\frac{V^{\varepsilon}-V}{\varepsilon}$. The ε parameter is not easily fit. A small parameter will lead to a less biased value estimation but the variance of the valuation increases the variance of this estimator. We take $\varepsilon = 0.005$ except when specified as our finite difference resolution parameter.

4.3.1 Fast Storage Results

We take a weekly product, a monthly product and a quarter product to achieve the comparisons. All figures give the hedging volume HV per day for the product used so that the volume associated to a given product with delivery length equal to n days is nHV. Figure 4 compares the three methods for the fourth week of July 2010. We observe that conditional tangent process and tangent process give very similar results. Finite difference gives results quite different from other methods but these results are unstable when changing ε parameter as shown on Fig. 5. Accuracy of each valuation in the finite difference procedure should be increased as ε decreases. Results obtained by conditional tangent process appears to be quite stable when changing the number of trajectories used in optimization as shown on Fig. 5. The Fig. 6 compares the three methods for December 2010. Notice that this product only appears the first of October. The Fig. 7 compares the three methods for the last quarter of 2010. Results for the monthly and quarter product results are very

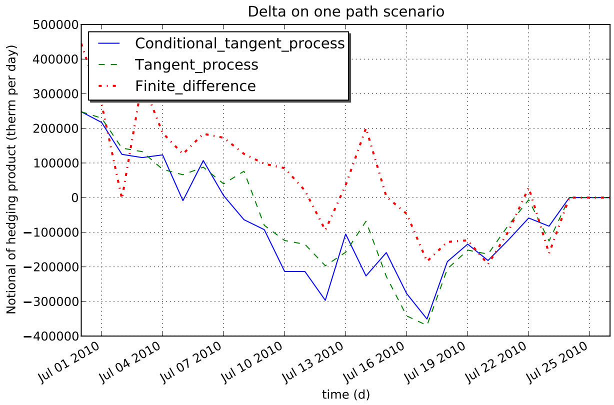
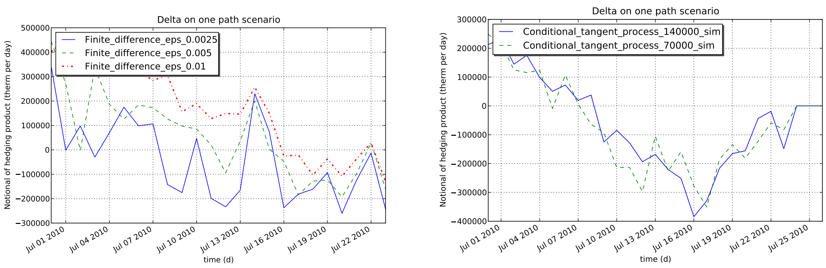


Fig. 4 Fast storage: Hedging strategy for the weekly product with delivery the last week of July 2010.



(a) Influence of ε parameter in finite difference (b) Influence of the number of simulations for the conditional tangent process

Fig. 5 Fast storage: Example of delta evolution for the weekly product with delivery the last week of July 2010.

similar for the three methods, finite difference still being little further from the two other methods. As for the time needed for each method calculating the hedge for all the products, the conditional delta took 10 minutes for the fast storage valuation and less than 20 seconds for the simulation of the hedging strategy on one scenario. As for tangent process, the simulation took 15 hours and the finite difference simulation took more than 4 days.

Besides this method comparison a simulation phase with 10000 simulations was achieved with the conditional delta method in less than 600 seconds. The standard deviation of the cash flow was reduced by a factor 5 from 8.21 10^8 to 1.64 10^8. Distribution of the cash flow with and without hedge is given in Fig. 8.

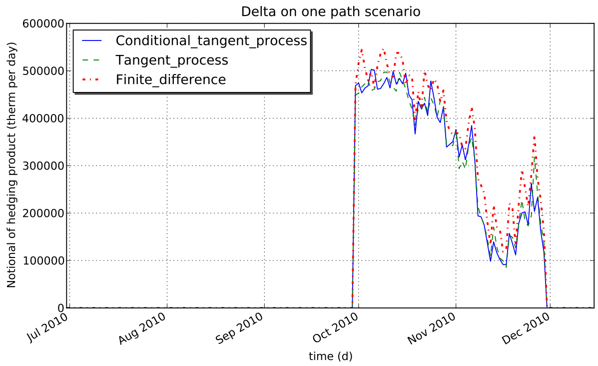


Fig. 6 Fast storage: Example of delta evolution for the monthly product December 2010.

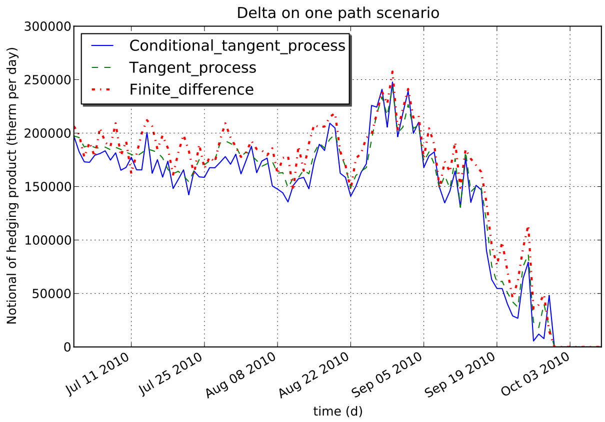


Fig. 7 Fast storage: Example of delta evolution for the quarter product Q4 2010.

4.3.2 Seasonal Storage Results

We took the same products as for the fast storage plus the year 2011. Results are given on figure 9, 10, 11, 12. The figure 10 compares the three methods for December 2010. The Fig. 11 compares tangent process and conditional tangent process for the 2010 last quarter product, Fig. 12 for the year 2011 product. Due to computational costs, results with finite difference were only available for the week and the monthly product. The convergence for the weekly product of the three methods to the same solution is more easily obtained for the seasonal storage than for the fast storage. On the different figures, we see a small kind of smoothing effect on the hedging strategy due to the conditional tangent process method not so obvious on the fast storage results. As for the time needed for each method, the conditional

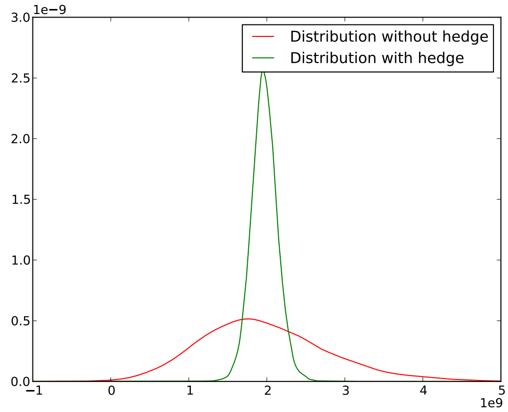


Fig. 8 Fast storage: cash flow distribution in pence with and without hedge computed with the conditional tangent method.

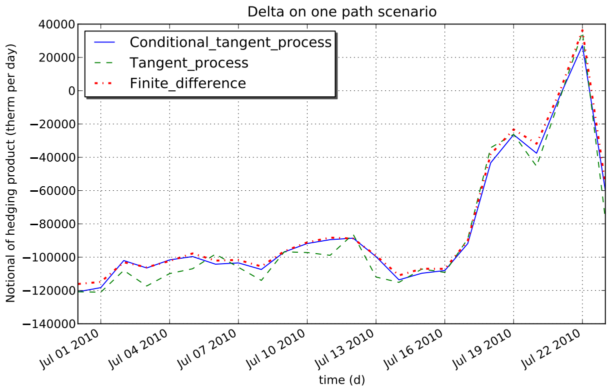


Fig. 9 Seasonal storage: Example of delta evolution for the weekly product with delivery the last week of July 2010.

delta took nearly two hours for valuation and 60 seconds for the simulation. As for tangent process, the simulation took 7 days for the seasonal storage and the hedge for all the products would have taken longer than a month for the finite difference method.
Once again a simulation phase with 10000 simulations taking 1400 seconds was achieved with the conditional delta method. It led to a reduction of the standard deviation of the results by more than a factor 8 from $9.60\ 10^8$ to $1.11\ 10^8$. Distribution of the cash flow with and without hedge is given in Fig. 13.

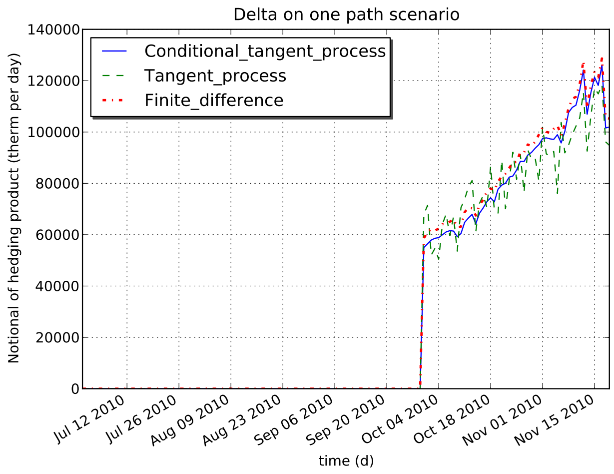


Fig. 10 Seasonal storage: Example of delta evolution for the monthly product December 2010.

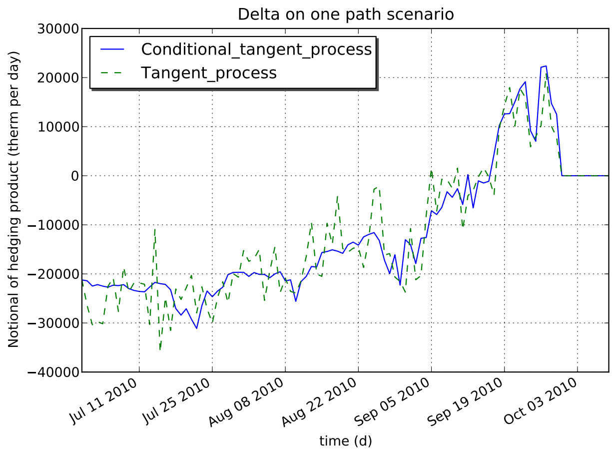


Fig. 11 Seasonal storage: Example of delta evolution for the quarter product Q4 2010 on one scenario

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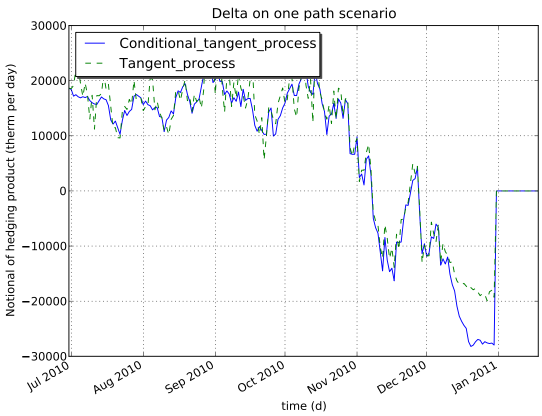

Fig. 12 Seasonal storage: Example of delta evolution for the year 2011 product.

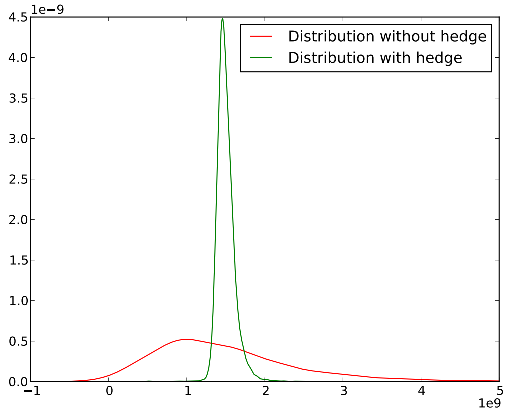

Fig. 13 Seasonal storage: cash flow distribution in pence with and without hedge computed with the conditional tangent method.

Algorithm 3 Valuate gas storage, calculate delta and conditional deltas

Require: Asset parameters
Ensure: Compute the storage value, the delta at time 0 and conditional deltas at each time step

// initialization
for $l = 0$ to L **do**
  $J^j_\kappa(l) = 0, D^j_\kappa(\kappa, l) = 0$, $j = 1$ to M
end for
// Backward recursion
for $i = \kappa - 1$ to 0 **do**
  for $l = 0$ to L **do**
    Calculate A^l_{out}, A^l_{in}, the maximum quantity of gas available for withdrawal, injection
    $PIn^j = -A^l_{in} S^j_i + \hat{\mathbb{E}}[J_{i+1}(l + A^l_{in}/\delta) | \, X_{t_i} = X^j_{t_i}]$ for $j = 1$ to M (injection)
    $PId^j = \hat{\mathbb{E}}[J_{i+1}(l) | \, X_{t_i} = X^j_{t_i}]$ for $j = 1$ to M (idle)
    $PW^j = A^l_{out} S^j_i + \hat{\mathbb{E}}[J_{i+1}(l - A^l_{out}/\delta) | \, X_{t_i} = X^j_{t_i}]$ for $j = 1$ to M (withdrawal)
    for $j = 1$ to M **do**
      if $PIn^j \geq PW^j$ and $PIn^j \geq PId^j$ **then**
        $\tilde{l} = l + A^l_{in}/\delta$
        $J^j_i(l) = -A^l_{in} S^j_i + J^j_{i+1}(\tilde{l})$
        $D^j_i(i, l) = -A^l_{in} Y^{t_i, j}_{t_i}$,
        for $m = i + 1$ to κ **do**
          $D^j_i(m, l) = D^j_{i+1}(m, \tilde{l})$
        end for
      else if $PId^j \geq PIn^j$ and $PId^j \geq PW^j$ **then**
        $J^j_i(l) = J^j_{i+1}(l)$
        $D^j_i(i, l) = 0$
        for $m = i + 1$ to κ **do**
          $D^j_i(m, l) = D^j_{i+1}(m, l)$
        end for
      else
        $\tilde{l} = l - A^l_{out}/\delta$
        $J^j_i(l) = A^l_{out} S^j_i + J^j_{i+1}(\tilde{l})$
        $D^j_i(i, l) = A^l_{out} Y^{t_i, j}_{t_i}$,
        for $m = i + 1$ to $\kappa - 1$ **do**
          $D^j_i(m, l) = D^j_{i+1}(m, \tilde{l})$
        end for
      end if
    end for
  end for
  Store the continuation value function $J(x, c_l) = \hat{\mathbb{E}}[J_{i+1}(l) | \, X_{t_i} = x]$ for all l
  Compute and store conditional delta for each delivery date $\hat{\Delta}(t_i, t_m, x, c_l) = \hat{\mathbb{E}}[D_i(m, l) | \, X_{t_i} = x]/Y^{t_m}_{t_i} \quad \forall m > i, \quad \forall l,$
end for
Storage value $\sum_{j=1}^{M} \frac{J^j_0(\tilde{l})}{M}$, $\tilde{l}$ initial index volume number
Delta for each delivery date $\sum_{j=1}^{M} \frac{D^j_0(m, \tilde{l})}{M} \quad \forall m > 0$

Algorithm 4 Cash flow simulation with and without hedging of the gas storage

Require: Asset parameters, continuation values J and conditional deltas $\hat{\Delta}$ calculated in optimization, see algorithm 3
Ensure: Calculate cash flow generated with or without hedging.
  c^j initialize volume at initial volume level for all $j = 1$ to M
  $v^j = 0$ initialize portfolio at zero for all $j = 1$ to M
  $v_h^j = 0$ initialize portfolio with hedge at zero for all $j = 1$ to M
  for $i = 0$ to $\kappa - 1$ **do**
    Get back the continuation value function $J(., c_l)$ at date t_i for all l
    Get back conditional delta function at date t_i : $\hat{\Delta}(t_i, t_m, ., c_l)$ for all l, $m > i$
    for $j = 1$ to M **do**
      for $p \in \mathcal{Q}_{t_i}$ // nest on future delivery period **do**
        Hedge for period $\mathcal{P}_p$
        $\Delta_p(t_i) = \frac{\sum_{t_k \in \mathcal{P}_p} \hat{\Delta}(t_i, t_k, X_{t_i}^j, c^j) F^j(t_i, t_k)}{\sum_{t_k \in \mathcal{P}_p} F^j(t_i, t_k)}$
        Integrated future product on delivery periode date i : $F_i = \sum_{t_k \in \mathcal{P}_p} F^j(t_i, t_k)$
        Integrated future product on delivery periode date $i+1$: $F_{i+1} = \sum_{t_k \in \mathcal{P}_p} F^j(t_{i+1}, t_k)$
        Hedge $v_h^j = v_h^j - (F_{i+1} - F_i)\Delta_p(t_i)$
      end for
      Calculate A_{out}, A_{in}, the maximum quantity of gas available for withdrawal, injection
      $PIn^j = -A_{in} S_{t_i}^j + J(X_{t_i}^j, c^j + A_{in})$ for $j = 1$ to M (injection)
      $PId^j = J(X_{t_i}^j, c^j)$ for $j = 1$ to M (idle)
      $PW^j = A_{out}^c S_{t_i}^j + J(X_{t_i}^j, c^j - A_{out})$ for $j = 1$ to M (withdrawal)
      for $j = 1$ to M **do**
        if $PIn^j \geq PW^j$ and $PIn^j \geq PId^j$ **then**
          // update volume level, portfolio with and without hedge if injection
          $v^j = -A_{in} S_{t_i}^j + v^j$
          $v_h^j = -A_{in} S_{t_i}^j + v_h^j$
          $c^j = c^j + A_{in}$
        else if $PW^j \geq PIn^j$ and $PW^j \geq PId^j$ **then**
          // update volume level, portfolio with and without hedge if withdrawal
          $v^j = A_{out} S_{t_i}^j + v^j$
          $v_h^j = A_{out} S_{t_i}^j + v_h^j$
          $c^j = c^j - A_{out}$
        end if
      end for
    end for
  end for

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