Optimal investment under relative performance concerns

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Joint work with N.Touzi

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Classical portfolio optimization: maximization of one’s utility with respect to one’s personal wealth or consumption

Economical literature: relative wealth concerns
Classical portfolio optimization: maximization of one’s utility with respect to one’s personal wealth or consumption

Economical literature: relative wealth concerns

Aim: Try to derive a portfolio optimization theory with such relative wealth concerns.
The market:

- a non-risky asset with 0 interest rate
- a $d$-dimensional risky asset $S$
- $N$ agents

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Optimization criterion for agent $i$:
- exponential utility function with risk sensitivity parameter $\eta > 0$
- relative performance sensitivity parameter $\lambda \in [0, 1]$
- average wealth of other agents $\bar{X}^i = \frac{1}{N-1} \sum_{j \neq i} X^j$
Thus agent $i$ wants to maximize upon admissible $\pi^i$:

$$-\mathbb{E}e^{-\eta[(1-\lambda)X^i_T + \lambda(X^i_T - \bar{X}^i_T)]]}$$

given other $\pi^j$ ($j \neq i$)
By symmetry, at the equilibrium, it is the same as:

$$\sup_{\pi^i} -\mathbb{E}e^{-\eta(1-\lambda)X^i_T}$$
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So the optimal portfolio is (for deterministic $\theta$, $\lambda < 1$):

$$\hat{\pi}_t^i = \frac{1}{\eta(1 - \lambda)} \sigma_t^{-1} \theta_t$$
- $|\hat{\pi}^i|$ is increasing in $\lambda$
- if $\lambda \to 1$, $|\hat{\pi}^i| \to \infty$ a.s.
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Define the market index:

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\bar{X}_T = \frac{1}{N} \sum_{i=1}^{N} X^i_T
\]
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Define the market index:

$$\bar{X}_T = \frac{1}{N} \sum_{i=1}^{N} X^i_T$$

At the equilibrium, its dynamics is given by:

$$d\bar{X}_t = \frac{1}{\eta(1 - \lambda)}[|\theta_t|^2 dt + \theta_t.dW_t]$$
Specific parameters:
- risk sensitivity parameter $\eta_i > 0$
- relative performance sensitivity parameter $\lambda_i \in [0, 1]$

Thus agent $i$ wants to maximize upon admissible $\pi_i$: 

$$-\mathbb{E}e^{-\eta_i[(1-\lambda_i)X^i_T+\lambda_i(X^i_T-\bar{X}^i_T)]}$$

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Portfolio constraints:

Each agent has an area of investment. \( \pi^i \) must stay in a certain \( A_i \) that will be assumed to be a vector sub-space of \( \mathbb{R}^d \).
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So finally we are looking for:

$$\sup_{\pi^i \in A_i} -\mathbb{E}e^{-\eta_i[X^i_T, \pi^i] - \lambda_i \bar{X}^i_T}$$
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So finally we are looking for:

$$\sup_{\pi^i \in A_i} -\mathbb{E} e^{-\eta_i [X^i_T, \pi^i - \lambda_i \bar{X}^i_T]}$$

And then look for Nash equilibria between the $N$ agents.
Using a result by Hu-Imkeller-Muller for optimal investment in incomplete markets, we can relate the single agent optimization problem with the following (quadratic) BSDE:

\[ dY_t^i = \left( \frac{|\theta_t|^2}{2\eta} - \frac{\eta}{2}|Z_t^i + \frac{\theta_t}{\eta} - P_{\sigma A_i}(Z_t^i + \frac{\theta_t}{\eta})|^2 \right) dt + Z_t^i dB_t \]

\[ Y_T^i = \lambda(\bar{X}_T^i - \bar{x}_i) = \frac{\lambda}{N-1} \sum_{j \neq i} \int_0^T \pi^j_u \sigma_u dB_u \]
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\begin{align*}
    dY^i_t &= \left( \frac{|\theta_t|^2}{2\eta} - \frac{\eta}{2} |Z^i_t + \frac{\theta_t}{\eta} P_{\sigma_t A_i} (Z^i_t + \frac{\theta_t}{\eta} )|^2 \right) dt + Z^i_t dB_t \\
    Y^i_T &= \lambda (\bar{X}^i_T - \bar{x}_i) = \frac{\lambda}{N-1} \sum_{j \neq i} \int_0^T \pi^j_u \sigma_u dB_u
\end{align*}
\]

And an optimal portfolio is given by:

\[
\sigma_t \hat{\pi}^i_t = P_{\sigma_t A_i} (Z^i_t + \frac{\theta_t}{\eta})
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And an optimal portfolio is given by:

\[ \sigma_t \hat{\pi}_t^i = P_{\sigma_t A_i}(Z_t^i + \frac{\theta_t}{\eta}) \]

Remark: there is no need for \( S \) to be a Markov process.
Putting them together it brings:

$$Y_0^i = - \frac{1}{\eta} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} + \frac{\eta}{2} \int_0^T |Q_t^i(Z_t^i)|^2 dt - \int_0^T (Z_t^i - \frac{\lambda}{N - 1} \sum_{j \neq i} P_t^j(Z_t^j)).dB_t$$

where $P_i$ is the orthogonal projection on $\sigma A_i$ and $Q_i = I - P_i$, $\mathbb{Q}$ is the martingale probability and $B$ a Brownian motion under $\mathbb{Q}$. 
After showing the regularity of the operator (under some assumptions), it can be rewritten as:

$$Y_0^i = -\frac{1}{\eta} \ln \frac{dQ}{dP} + \frac{\eta}{2} \int_0^T |Q_t^i([\psi_t(\zeta_t)]^i)|^2 dt - \int_0^T \zeta_t^i dB_t$$

where $Y \in \mathbb{R}^N$, $\zeta \in M_{N,d}(\mathbb{R})$ and $\psi \in GL(M_{N,d}(\mathbb{R}))$. 
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→ $N$-dimensional system of coupled quadratic BSDEs.
Assume the following:

\[
\prod_{i=1}^{N} \lambda_i < 1 \quad \text{or} \quad \bigcap_{i=1}^{N} A_i = \{0\}
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**Theorem:** There exists a unique equilibrium and an optimal portfolio for agent \(i\) is given by:

\[
\pi^i = \frac{1}{\eta} \sigma^{-1} P_i \left( [I - \frac{\lambda}{N-1}] \left( \sum_{j \neq i} P_j \right) \left( I + \frac{\lambda}{N-1} P_i \right)^{-1} \theta \right)
\]

\((P_i\) is the orthogonal projection on \(\sigma A_i)\)
In the simple case where $d$ is fixed we have:

**Theorem:** Let $d$ be fixed, and assume moreover that

$$\frac{1}{N} \sum_{i=1}^{N} P_i \rightarrow U \text{ in } \mathcal{L}(\mathbb{R}^d) \text{ with } \|\lambda U\| < 1.$$  

Then $\pi^i_N \rightarrow \pi^i_\infty$ uniformly where:

$$\pi^i_\infty = \frac{1}{\eta} \sigma^{-1} P_i[(I - \lambda U)^{-1} \theta]$$
Once again the market index is: 

$$\bar{X}_t^N = \frac{1}{N} \sum_{i=1}^{N} X_t^i$$

And we find:

$$d\bar{X}_t^\infty = \frac{1}{\eta} U(I - \lambda U)^{-1} \theta_t \cdot [\theta_t dt + dW_t]$$
Once again the market index is: \( \bar{X}_t^N = \frac{1}{N} \sum_{i=1}^{N} X_t^i \)
And we find:

\[
d\bar{X}_t^{\infty} = \frac{1}{\eta} U(I - \lambda U)^{-1} \theta_t \left[ \theta_t dt + dW_t \right]
\]

Moreover, \( U(I - \lambda U)^{-1} \) is diagonalizable with eigenvalues

\[
0 < \frac{\mu_1}{1 - \lambda \mu_1} < \cdots < \frac{\mu_d}{1 - \lambda \mu_d} < 1
\]

and with the same orthonormal eigenvectors as \( U \) (independent of \( \lambda \)).
→ The risk (volatility) of the market increases with $\lambda$. 
Each agent can invest in the whole market:

$$\forall i, \ A_i = \mathbb{R}^d$$
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Under the assumption $\prod_{j=1}^{N} \lambda_j < 1$, there is a unique equilibrium.
First case: $\forall i$, $\lambda_i = \lambda$, then:

$$\hat{\pi}_t^i = \left[ \frac{N - 1}{N + \lambda - 1} + \frac{\lambda N}{(1 - \lambda)(N + \lambda - 1)} \frac{\eta_i}{\eta^N} \right] \pi_{t,0}^i$$

$\eta^N$ is the harmonic average of the $\eta^i$. 
As \( N \to \infty \), if \( \eta^N \to \eta > 0 \) then the equilibrium portfolio of agent \( i \) converges uniformly to:

\[
\hat{\pi}_{t,^\infty,i} = \left(1 + \frac{\lambda}{1 - \lambda \eta} \right) \pi_{t,^0,i}
\]
As $N \to \infty$, if $\eta^N \to \eta > 0$ then the equilibrium portfolio of agent $i$ converges uniformly to:

$$\hat{\pi}_t^{\infty,i} = (1 + \frac{\lambda}{1 - \lambda} \frac{\eta_i}{\eta})\pi_0^{0,i}$$

Same conclusions as in the beginning.
Second case: $\forall j \neq i_0$, $\lambda_j = 1$, $\lambda_{i_0} < 1$ ($\forall i$, $\eta_i = \eta$), then:

$$\hat{\pi}_{i_0}^t = \left[ \frac{1}{1 - \lambda_{i_0}} + \frac{\lambda_{i_0}(N - 1)}{1 - \lambda_{i_0}} \right] \pi_0^t$$

$\rightarrow$ Impact of surrounding "stupidity".
Second case: $\forall j \neq i_0, \lambda_j = 1, \lambda_{i_0} < 1 \ (\forall i, \eta_i = \eta)$, then:

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As $N \to \infty$, even if $\lambda_{i_0} < 1$, $|\pi^i_t| \to \infty \ a.s \ (\text{except for } \lambda_{i_0} = 0)$. 
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$\rightarrow$ Impact of surrounding "stupidity".
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\begin{itemize}
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  \item $\sigma^2 = \sigma^2 \begin{pmatrix} 1 & \rho^2 \\ \rho^2 & 1 \end{pmatrix}$ with $\rho \in (-1, 1)$ and $\sigma > 0$
  \item we also assume $\forall i$, $\theta_i = \theta$.
\end{itemize}
As $N \to \infty$ we find:

\[
\hat{\pi}_i = \frac{\theta}{\eta \sigma \frac{1}{1 - \lambda \rho^2}} e_i
\]

So:

- the more you look at other agents ($\lambda$ close to 1)
- the more correlated the assets are ($\rho$ close to 1)
- the more risk you take.

For independent investments ($\rho = 0$), we find the classical optimal portfolio: no impact of $\lambda$. 

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\[
\begin{align*}
\sigma &= \sigma I & \forall i, \theta_i &= \theta.
\end{align*}
\]
We find:

$$\hat{\pi}_t^i = \frac{\theta}{\eta\sigma} \frac{1}{1 - \lambda + \frac{\lambda}{N-1}} \sum_{j \neq i} e_j$$

Same kind of conclusions as for investment on the whole market, but smaller impact of $\lambda$, especially for small $N$. 

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Short Bibliography

Special thanks to J. Lebuchoux - Reech Aim