Swing Options Valuation: a BSDE with Constrained Jumps Approach

M. Bernhart, H. Pham, P. Tankov and X. Warin

Abstract We introduce a new probabilistic method for solving a class of impulse control problems based on their representations as Backward Stochastic Differential Equations (BSDEs for short) with constrained jumps. As an example, our method is used for pricing Swing options. We deal with the jump constraint by a penalization procedure and apply a discrete-time backward scheme to the resulting penalized BSDE with jumps. We study the convergence of this numerical method, with respect to the main approximation parameters: the jump intensity λ, the penalization parameter $p > 0$ and the time step. In particular, we obtain a convergence rate of the error due to penalization of order $(\lambda p)^{\alpha-\frac{1}{2}}, \forall\alpha \in \left(0, \frac{1}{2}\right)$. Combining this approach with Monte Carlo techniques, we then work out the valuation problem of (normalized) Swing options in the Black and Scholes framework. We present numerical tests and compare our results with a classical iteration method.

Keywords Backward stochastic differential equations with constrained jumps, Impulse control problems, Swing options, Monte Carlo methods

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1 Introduction

In this report, we introduce a new probabilistic method for solving impulse control problems based on their representations as Backward Stochastic Differential Equations (BSDEs for short) with constrained jumps. As an example, our method is used for pricing Swing options in the Black and Scholes framework.

BSDEs provide alternative characterizations of the solution to multiple-obstacle, optimal switching (see among others [19, 8, 20, 26, 13]) and more generally impulse control problems: Kharroubi et al. [21] recently introduced a family of BSDEs with constrained jumps providing a representation of the solution to such problems. A challenging question is that of the numerical approximation of this kind of BSDEs with constrained jumps.

A discrete-time backward scheme for solving BSDEs with jumps (without constraint) has been introduced by Bouchard and Elie [3]. In our case, the main difficulty comes from the constraint, which concerns the jump component of the solution. These BSDEs do not a priori involve any Skorohod type minimality condition. In consequence, classical approaches by projected schemes (discretely reflected backward schemes) used for example by [2] and [10] cannot be used. In particular, these latter authors introduce a discretely obliquely reflected numerical scheme for solving optimal switching problems and obtain a convergence rate of order $|\pi|^{\frac{1}{2}-\varepsilon}, \forall \varepsilon > 0$ for a time step equal to $|\pi|$. However, this result holds in a no-jump case where the forward process is uncontrolled.

We consider a penalization procedure to deal with the constraint on jumps and provide a convergence rate of the penalized solution to the exact solution. This allows us to establish a convergence rate of the error between the solution of the considered impulse control problem and the numerical approximation given by the discrete-time solution to the penalized BSDE with jumps, as the penalization coefficient and the number of time steps go to infinity.

The rest of the report is structured as follows: in Section 2, we set the considered impulse control problem in the mathematical framework of BSDEs with constrained jumps. We present in Section 3 our penalization approach and provide a global convergence rate of our approximation. In Section 4, our method is used for pricing multi-exercise options, so-called (normalized) Swing options. This multiple optimal stopping time problem leads to a particularly degenerate three-dimensional impulse control problem. We combine our BSDE-based approach with Monte Carlo techniques and deal with Swing options with a small maximal number of exercises rights, due to large computational times. We compare our pricing results with those obtained by a classical iteration-based approach proposed for example by [9].

2 BSDE Representation for Impulse Control Problems

Let T be a given time horizon. We work in a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$, on which is defined a d-dimensional Brownian motion W and a Poisson process N with intensity $\lambda > 0$. We denote by $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$, the augmentation of the natural filtration generated by W and N, by $\mathbb{F}^W = (\mathscr{F}_t^W)_{t\geq 0}$ the one generated by W, and by $\mathscr{P}$, the σ-algebra of predictable sub-sets of $\Omega \times [0,T]$.

Notation

Throughout this report, the euclidean norm defined on $\mathbb{R}^d$ or on $\mathbb{R}$ will be indiscriminately denoted by $|\cdot|$. The matrix transposition is denoted by $\perp$. In addition, unless specified otherwise, C will denote a strictly positive constant depending only on Lipschitz constants of the coefficients of the problem, see assumptions (H) and (H′) below, and constants T, $|b(0)|$, $|\sigma(0)|$, $|\gamma(0)|$, $|f(0)|$, $|\kappa(0)|$ and $|g(0)|$.

Besides, we shall use the standard notations:

- $\mathscr{S}^2$, the set of real-valued càdlàg adapted processes $Y = (Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathscr{S}^2} := \left(\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right]\right)^{\frac{1}{2}} < \infty\,.$$

- $\mathscr{A}^2$, the sub-set of $\mathscr{S}^2$ such that

$$\mathscr{A}^2 := \left\{K \in \mathscr{S}^2 : (K_t)_{0\leq t\leq T} \text{ nondecreasing }, K_0 = 0\right\}\,.$$

- $\mathrm{L}^2_{\mathbb{F}}([0,T])$, the set of real-valued adapted processes $(\phi_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^2\, dt\right] < \infty\,.$$

- $\mathrm{L}^2(W)$, the set of real-valued $\mathscr{P}$-measurable processes $Z = (Z_t)_{s\leq t\leq r}$ such that

$$\|Z\|_{\mathrm{L}^2(W)} := \left(\mathbb{E}\left[\int_0^T |Z_t|^2\, dt\right]\right)^{\frac{1}{2}} < \infty\,.$$

- $\mathrm{L}^2(N)$, the set of real-valued $\mathscr{P}$-measurable processes $V = (V_t)_{s\leq t\leq r}$ such that

$$\|V\|_{\mathrm{L}^2(N)} := \left(\mathbb{E}\left[\int_0^T |V_t|^2\, \lambda\, dt\right]\right)^{\frac{1}{2}} < \infty\,.$$

- $\mathscr{V}$ denotes the set of $\mathscr{P}$-measurable essentially bounded processes, valued in $(0,\infty)$ and $\mathscr{V}^p = \left\{\nu^p \in \mathscr{V} : \nu_t^p \leq p \text{ a.s.}\right\}$.

2.1 A Class of Impulse Control Problems

We consider the class of impulse control problems whose value function is defined by:

$$v(t,x) = \sup_{u=(\tau_k)_{k\geq 1}\in\mathscr{U}_{(t,T]}} \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u})ds + \sum_{\substack{k\geq 1\\ t<\tau_k\leq T}} \kappa(X_{\tau_k^-}^{t,x,u}) \right]. \quad (1)$$

An impulse strategy $u = (\tau_k)_{k\geq 1}$ is said to be admissible for problem (1) (and belongs to $\mathscr{U}_{(t,T]}$) if it is a non-decreasing sequence of $\mathbb{F}^W$-stopping times valued in $(t,T]$ (we set by convention $\tau_0 = t$) such that, if

$$n_{(t,T]}^u := \sharp\{k \geq 1 : t < \tau_k \leq T\}$$

denotes the (random) number of interventions of the strategy u before time T, then

$$\mathbb{E}\left|n_{(t,T]}^u\right|^2 < C, \quad (2)$$

for some universal constant $C > 0$. The controlled state variable $X^{t,x,u}$ is a càdlàg process with dynamics

$$X_s^{t,x,u} = x + \int_t^s b(X_r^{t,x,u})dr + \int_t^s \sigma(X_r^{t,x,u})dW_r + \sum_{t<\tau_k\leq s} \gamma(X_{\tau_k^-}^{t,x,u}), \quad \forall s \geq t. \quad (3)$$

Between two successive intervention times τ_k and τ_{k+1}, the state variable evolves as a diffusion process and the controller makes an integral profit f. At each decided intervention time τ_k, he gives an impulse to the system: the state process jumps with a size $X_{\tau_k}^u - X_{\tau_k^-}^u = \gamma(X_{\tau_k^-}^u)$ and he obtains the intervention gain κ.

We consider standard assumptions on the coefficients of the problem:

(H) $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $\gamma : \mathbb{R}^d \mapsto \mathbb{R}^d$ are Lipschitz continuous and γ is uniformly bounded.
$f : \mathbb{R}^d \mapsto \mathbb{R}$, $\kappa : \mathbb{R}^d \mapsto \mathbb{R}$ and $g : \mathbb{R}^d \mapsto \mathbb{R}$ are Lipschitz continous.

(H′) The maps b, σ, γ, and g belong to $\mathscr{C}_b^1(\mathbb{R}^d)$ and have Lipschitz continuous derivatives.

A straightforward computation using (H), (2) and Gronwall's lemma shows that

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d, \quad \sup_{u\in\mathscr{U}_{(t,T]}} \mathbb{E}\left[\sup_{t\leq s\leq T} \left|X_s^{t,x,u}\right|^2\right] < \infty. \quad (4)$$

Finally, we will assume the existence of an optimal strategy $u^* = (\tau_k^*)_{k\geq 1} \in \mathscr{U}_{(t,T]}$ to problem (1). We refer for example to [4] and [25] in the infinite horizon case, for specific conditions on the coefficients of the problem which ensures such an existence.

2.2 Link to BSDEs with Constrained Jumps

Let us consider the BSDE with constrained jumps

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s - \int_t^T V_s dN_s + \int_t^T dK_s, \quad \forall 0 \le t \le T \\ V_t + \kappa(X_{t^-}) \le 0, \quad \forall 0 \le t \le T \end{cases} \tag{5}$$

where X is the (uncontrolled) jump diffusion process with dynamics

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \gamma(X_{t^-})dN_t. \tag{6}$$

Under (H), this SDE admits an unique solution in $\mathscr{S}^2$ and it is shown in [21] that under the additional assumption (H_1) given below, (5) admits a unique *minimal* solution $(Y,Z,V,K) \in \mathscr{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N) \times \mathscr{A}^2$ with K predictable. We refer to [21] for sufficient conditions for (H_1) and (H_1').

The solution (Y,Z,V,K) is said to be *minimal* if and only if it has the smallest component Y in the (infinite) class of solutions to (5). $(Y_t)_{t\ge 0}$ is called the value process and jumps with a size $V_t = Y_t - Y_{t^-}$.

(H_1) There exists a solution $(\tilde{Y},\tilde{Z},\tilde{K}) \in \mathscr{S}^2 \times \mathrm{L}^2(W) \times \mathscr{A}^2$ to

$$Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s + \int_t^T \kappa(X_{s^-})dN_s + \int_t^T dK_s.$$

(H_1') (H_1) holds and $\tilde{Y}_t = \tilde{v}(t,X_t), \forall 0 \le t \le T$ for some $\tilde{v}$ with linear growth.

(H_2) There exists a non negative function $\varphi \in \mathscr{C}^2(\mathbb{R}^d)$ and a constant $\rho > 0$ s.t.

$$\begin{array}{ll} \mathscr{L}\varphi + f \le \rho\varphi, & \varphi - \mathscr{H}\varphi > 0, \\ \varphi \ge g, & \lim_{|x|\to\infty} \frac{\varphi(x)}{1+|x|} = \infty, \end{array}$$

in which $\mathscr{L}$ is the local component of the generator of the process X and $\mathscr{H}$, the intervention operator:

$$\begin{aligned} \mathscr{L}v(t,x) &= b(x)\cdot D_x v(t,x) + \frac{1}{2}\mathrm{Tr}\left(\sigma\sigma^{\perp}(x)D_x^2 v(t,x)\right), \\ \mathscr{H}v(t,x) &= v(t,x+\gamma(x)) + \kappa(x)\ . \end{aligned}$$

Let $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t\le s\le T}$ be the solution to (5) when $X \equiv (X_s^{t,x})_{t\le s\le T}$ is the solution starting at x in t to SDE (6). Under assumptions (H), (H_1') and (H_2), [21] show that the solution to impulse control problem (1) coincides with initial value of component $Y^{t,x}$:

$$Y_t^{t,x} = v(t,x) \tag{7}$$

and is equal to the (unique) solution with linear growth to quasi-variationnal inequality

$$\begin{aligned} \min\Big\{-\tfrac{\partial v}{\partial t}(t,x) - \mathscr{L}v(t,x) - f(t,x)\,;\\ v(t,x) - \mathscr{H}v(t,x)\big\} &= 0,\ \forall (t,x) \in [0,T) \times \mathbb{R}^d, \\ \min\{v(T^-,x) - g(x); v(T^-,x) - \mathscr{H}v(T^-,x)\} &= 0,\ \forall x \in \mathbb{R}^d. \end{aligned} \tag{8}$$

Let us mention that in general, the terminal condition $v(T^-,\cdot) = g$ is irrelevant, because of the possible discontinuity of Y in T^- due to constraints: the relaxed terminal condition in (8) expresses the possibility of a jump at time T^-.

Remark 1. For a better intuition, the following interpretation to solution (Y,Z,V,K) holds when assuming $v \in \mathscr{C}^{1,2}([0,T],\mathbb{R}^d)$:

$$\begin{aligned} \forall 0 \le t \le T,\ Y_t &= v(t,X_t) \\ Z_t &= \sigma(t,X_{t^-})D_x v(t,X_{t^-}) \\ V_t &= v(t,X_{t^-} + \gamma(X_{t^-})) - v(t,X_{t^-}) \\ &= \mathscr{H}v(t,X_{t^-}) - v(t,X_{t^-}) - \kappa(X_{t^-}) \\ K_t &= \int_0^t \left(-\frac{\partial v}{\partial t} - \mathscr{L}v - f\right)(s,X_s)ds. \end{aligned}$$

The constraint in (5) means thus that the obstacle condition is satisfied, namely $v(t,X_{t^-}) - \mathscr{H}v(t,X_{t^-}) \ge 0$.

3 Convergence of the Numerical Approximation by Penalization

It does not seem possible to use the minimality condition of the solution to BSDE with constrained jumps (5) directly in a numerical scheme. We thus propose an approach by penalization of the jump constraint. The penalized constraint is introduced in the BSDE driver: when the constraint is fulfilled, this penalization term disappears, and otherwise penalizes the driver with an exploding factor p.

In Theorem 1, we provide an explicit rate of convergence of our approximation with respect to the parameters introduced: namely, the jump intensity λ, the penalization coefficient p and the time step. Such an error estimate is essential for numerical purposes (understanding of the numerical impact of those parameters) and allows to adjust in practice the fineness of the time grid in relation to (λ, p).

3.1 Approximation by Penalization

Given a parameter value $p > 0$, the penalized BSDE is:

$$Y_t^p = g(X_T) + \int_t^T \left[f(X_s) + p\,(V_s^p + \kappa(X_{s^-}))^+ \lambda \right] ds \\ - \int_t^T Z_s^p dW_s - \int_t^T V_s^p dN_s, \quad \forall 0 \le t \le T \tag{9}$$

which admits an unique solution $(Y^p, Z^p, V^p) \in \mathscr{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ from the classical theory of BSDEs with jumps. In addition, the sequence of penalized solutions $(Y^p, Z^p, V^p)_p$ tends in $\mathrm{L}^2_{\mathbb{F}}([0,T]) \times \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ to the minimal solution (Y, Z, V) to (5) as p goes to infinity, see [21]. Besides, the convergence of $(Y^p)_p$ to Y is monotone and increasing.

Let $(Y^{p,t,x}, Z^{p,t,x}, V^{p,t,x})$ be the solution to (9) when $X \equiv (X_s^{t,x})_{t \le s \le T}$. We consider the following error introduced by this penalization procedure:

$$\mathscr{E}^p := \sup_{0 \le t \le T} \left| v(t,x) - Y_t^{p,t,x} \right|. \tag{10}$$

For any $t < \eta \le T$, let us introduce

$$v_T^\eta(t,x) = \sup_{u=(\tau_k)_{k\ge 1} \in \mathscr{U}_{(t,T-\eta]}} \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u}) ds + \sum_{\substack{k \ge 1 \\ t < \tau_k \le T}} \kappa(X_{\tau_k^-}^{t,x,u}) \right] \tag{11}$$

which corresponds to initial problem (1) restricted to the sub-set of strategies taking values in $(t, T-\eta]$. We shall denote by $u^{\eta *} = (\tau_k^{\eta *})_{k \ge 1}$ an $\eta^{\frac{1}{2}}$-optimal strategy to problem (11) (the existence of an optimal strategy is not ensured) and by $n^{\eta *}$ the number of impulses in strategy $u^{\eta *}$ that is:

$$n^{\eta *} := \sharp \left\{ k \ge 1 : t < \tau_k^{\eta *} \le T - \eta \right\}.$$

We will use the following additional assumptions:

(H^n) There exists some $\bar{n} \in \mathbb{N}^*$ such that

$$\forall j \ge \bar{n}, \quad \mathbb{P}(n^{\eta *} \ge j) \le l(j)$$

for some map l such that $l(j) \le e^{-Cj}$ for some some constant $C > 0$.

(H*) There exists a map h such that $h(\varepsilon) = \mathscr{O}_{\varepsilon \to 0}(\varepsilon^{\frac{1}{2}})$ and

$$\forall \varepsilon > 0, \quad \mathbb{P}\left(\min_{k \ge 1} \left| \tau_{k+1}^{\eta *} - \tau_k^{\eta *} \right| \le \varepsilon \right) \le h(\varepsilon).$$

Remark 2 (Assumptions (H^n) *and* (H^*)*).* Both assumptions (H^n) and (H^*) are directly satisfied for the problem of Swing options valuation since the number of exercises right is almost surely bounded by some $n_{\max}$ and there is some fixed time delay $\delta > 0$ between two consecutive interventions.

More generally, (H^n) is intuitively satisfied as soon as the controlled state variable is constrained almost surely and admits jumps of constant sign, see Example 1. (H^*) is verified for sufficiently smooth problems, see for example the case of optimal forest management studied in [1] and in particular Proposition 5.1.1 p. 140.

Example 1. Let us assume that the state variable defined in (3) is such that

- b is uniformly bounded and $\sigma > 0$ constant,
- for some constant $c > 0$,

$$\sup_{x\in\mathbb{R}^d} \gamma(x) \leq -c,$$

and that the optimal strategy u^* implies $X_T^{u^*} \geq 0$ a.s. Then a straightforward computation shows that the (random) number $n^*_{(0,T]}$ of optimal impulses before time T satisfies, for any $a > 0$,

$$\mathbb{P}\left(n^*_{(0,T]} > n\right) = \mathscr{O}\left(e^{-an}\right) \quad \text{as } n \to +\infty.$$

Proposition 1. *Let assumptions* (H)*,* (H^n)*,* (H^*)*,* (H'_1) *and* (H_2) *be satisfied. Then the penalization error in* (10) *admits the following bound as p goes to infinity:*

$$\mathscr{E}^p \leq C\left(\frac{\bar{n}\bar{C}^{\bar{n}}}{(\lambda p)^{\frac{1}{2}-\alpha}}\right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right).$$

for some constants $C > 0$ *and* $\bar{C} > 1$*, which do not depend either on* λ*, p,* $\bar{n}$ *or* α*.*

Proof. We provide the main arguments of the proof and refer the reader to [1] pp. 81-98 for more details. The main idea comes from the following explicit functional representation available for $Y^{p,t,x}$, see [21]:

$$Y_t^{p,t,x} = \operatorname*{ess\,sup}_{\nu^p\in\mathscr{V}^p} \mathbb{E}^{\nu^p}\left[g(X_T^{t,x}) + \int_t^T f(X_s^{t,x})ds + \int_t^T \kappa(X_{s^-}^{t,x})dN_s\right], \qquad (12)$$

where $\mathbb{E}^{\nu^p}$ denotes the expectation under the probability measure $\mathbb{P}^{\nu^p}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathscr{F}_T)$ with Radon-Nikodym density

$$\left.\frac{d\mathbb{P}^{\nu^p}}{d\mathbb{P}}\right|_{\mathscr{F}_T} = e^{-\int_0^T(\nu_s^p-1)\lambda ds} e^{\int_0^T \ln(\nu_s^p)dN_s}.$$

The specificity of such a change of measure is that it impacts only the jump parts of the processes: under $\mathbb{P}^{\nu^p}$, the Brownian motion W remains unchanged whereas N has a (stochastic) intensity $(\lambda\nu_s^p)_{s\geq 0}$. We shall denote by N^p, the doubly stochastic

Poisson process (Cox process) with intensity $(\lambda \nu_s^p)_{s\geq 0}$ under $\mathbb{P}$, by $(\tau_k^p)_{k\geq 1}$ the sequence of its jump dates and by X^p the solution to (6) driven by N^p.

In view of (11) and (12), we introduce a convenient measure change, which intuitively forces the penalized solution to jump as soon as possible after that an optimal impulse occurs:

$$\forall s \geq 0, \quad \nu_s^p = \begin{cases} p & \text{if } \sum_{k\geq 1} \mathbb{1}_{\left\{\tau_k^{\eta*} < s\right\}} \neq N_s^p, \\ 0 & \text{else.} \end{cases} \tag{13}$$

Notice that ν^p is a $\mathscr{P}$-measurable process bounded by p a.s. By definition of the counting process N^p,

$$\forall s \geq 0, \quad \mathbb{P}\left(\tau_k^p - \tau_k^{\eta*} > s \mid \sum_{j\geq 1} \mathbb{1}_{\left\{\tau_j^p \leq \tau_k^{\eta*}\right\}} = k-1\right) = e^{-\lambda p s}. \tag{14}$$

In other words, the increment $\tau_k^p - \tau_k^{\eta*}$ has an exponential distribution with parameter (λp), conditionally to the fact that X^p has jumped one time less than $X^{u^{\eta*}}$. Denoting by

$$\tilde{Y}_t^{p,t,x} := \mathbb{E}\left[g(X_T^{t,x,p}) + \int_t^T f(X_s^{t,x,p})ds + \sum_{\substack{k\geq 1 \\ t<\tau_k^p \leq T}} \kappa(X_{\tau_k^{p-}}^{t,x,p})\right], \tag{15}$$

one can show by using the continuity of the value function in its maturity variable (see Proposition 2.3.1 in [1] p. 78), that the penalization error is related to the distance between (11) and (15) by

$$\mathscr{E}^p \leq C\eta^{\frac{1}{2}} + \sup_{0\leq t\leq T} \left|v_T^\eta(t,x) - \tilde{Y}_t^{p,t,x}\right|,$$

see Lemma 3.1.2 in [1] p. 83. As p goes to infinity, the jump diffusion $X^{t,x,p}$ tends to mimic the dynamics of controlled state variable $X^{t,x,u^{\eta*}}$, since each τ_k^p becomes closer to its corresponding $\tau_k^{\eta*}$, recall (14). Then, an iteration on the indices of jump dates $(\tau_k^{\eta*})_{k\geq 1}$ allows to compute an estimate for

$$\sup_{0\leq t\leq T} \left|v_T^\eta(t,x) - \tilde{Y}_t^{p,t,x}\right|.$$

We conclude the proof by choosing an appropriate η with respect to (λ, p), see Theorem 3.2.1 in [1] p. 89. □

3.2 Convergence Rate of the Numerical Scheme

Given a regular time grid $\pi = \{t_0 = 0, t_1, \dots, t_N = T\}$, we assume that the solution X to (6) can be simulated on π either perfectly or by using an Euler scheme and denote its discrete-time version by X^π. Along the lines of [3], we consider the following backward discrete-time scheme for numerically solving the penalized BSDE with jumps (9),

$$\begin{cases} Y^{p,\pi}_{t_N} = g(X^\pi_{t_N}) \\ \forall t_n \in \pi, t_n < T : \\ \quad V^{p,\pi}_{t_n} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} \Delta \tilde{N}_{t_{n+1}} | \mathscr{F}_{t_n} \right] \\ \quad Z^{p,\pi}_{t_n} = \frac{1}{\Delta t_{n+1}} \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} \Delta W_{t_{n+1}} | \mathscr{F}_{t_n} \right] \\ \quad Y^{p,\pi}_{t_n} = \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} | \mathscr{F}_{t_n} \right] \\ \qquad\qquad + \left[f(X^\pi_{t_n}) + \left(p \left(V^{p,\pi}_{t_n} + \kappa(X^\pi_{t_n}) \right)^+ - V^{p,\pi}_{t_n} \right) \lambda \right] \Delta t_{n+1} \end{cases} \tag{16}$$

where $\Delta t_{n+1} = t_{n+1} - t_n$, $\Delta W_{t_{n+1}}$ is the Brownian increment on $[t_n, t_{n+1}]$ and $\Delta \tilde{N}_{t_{n+1}}$ the compensated version of the Poisson increment $\Delta N_{t_{n+1}}$ on $[t_n, t_{n+1})$.

We consider the classical discretization error between the continuous-time solution (Y^p, Z^p, V^p) in (9) and its discrete-time approximation $(Y^{p,\pi}, Z^{p,\pi}, V^{p,\pi})$ in (16), that is

$$\mathscr{E}^\pi(Y^p) := \left(\max_{n < N-1} \sup_{t_n \le t \le t_{n+1}} \mathbb{E}\left| Y^p_t - Y^{p,\pi}_{t_n} \right|^2 \right)^{\frac{1}{2}}$$

$$\mathscr{E}^\pi(Z^p) := \left(\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left| Z^p_t - Z^{p,\pi}_{t_n} \right|^2 dt \right)^{\frac{1}{2}}$$

$$\mathscr{E}^\pi(V^p) := \left(\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left| V^p_t - V^{p,\pi}_{t_n} \right|^2 \lambda dt \right)^{\frac{1}{2}}.$$

Because of the lack of first order regularity of the driver of the penalized BSDE

$$f^p(x,v) := f(x) + \left(p(v + \kappa(x))^+ - v \right) \lambda, \quad \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}, \tag{17}$$

classical regularization arguments for the FBSDE coefficients and Malliavin differentiation representations allow us to provide an explicit convergence rate of order $|\pi|^{\frac{1}{2}}$ for errors $\mathscr{E}^\pi(Y^p)$ and $\mathscr{E}^\pi(V^p)$, but only of order $|\pi|^{\frac{1}{4}}$ for error $\mathscr{E}^\pi(Z^p)$, see Proposition 2.

The impact of the penalization coefficient p on the convergence of backward discrete-time schemes is well known in practice, even if there does not exist, to our best knowledge, any explicit computation in the literature (see for example the numerical experiments of [23] for the resolution by penalization of a BSDE with one reflecting barrier). Basically, as p increases at fixed discrete-time step, the quantity $f^p(\cdot)\Delta t_{n+1}$ explodes, leading to a numerical explosion of the approximate values $Y^{p,\pi}$, see (16).

We show rigorously in Proposition 2 that the discretization error grows exponentially with (λp^2). This is due to the linear dependence in λ and p of the BSDE driver f^p, see (17), and estimate computations based on the use of Gronwall's lemma.

Proposition 2. *Assume* (H). *Then, as soon as*

$$\exists C > 0, \quad |\pi| \leq \frac{C}{\lambda p^2}, \tag{18}$$

we get the following bounds as p goes to infinity:

$$\mathscr{E}^\pi(Y^p) \leq C\left((1+\lambda)^2 \lambda p \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{2}}\right)$$
$$\mathscr{E}^\pi(V^p) \leq C\left((1+\lambda)^2 \lambda^{\frac{3}{2}} p^2 \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{2}}\right).$$

Under (H′)*, there exists a version of Z^p such that,*

$$\mathscr{E}^\pi(Z^p) \leq C\left((1+\lambda)^2 \lambda^{\frac{5}{2}} p^3 \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{4}}\right),$$

for some constants $C > 0$ and $\bar{C} > 1$, which do not depend either on λ, p or $|\pi|$.

Proof. This follows from the same arguments as [15]: computations using Itô and Gronwall's lemma and regularization and Malliavin differentiation arguments applied to the penalized BSDE with jumps (9). We refer the reader to Corollary 4.4.1 in [1] p. 117 for a detailed proof. □

Propositions 1 and 2 enable us to establish a global convergence rate of the error introduced by our approximation by penalization.

Theorem 1. *Let the assumptions of Proposition 1 be satisfied. Then*

$$\mathscr{E}^p + \mathscr{E}^\pi(Y^p) \leq C\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} + (1+\lambda)^2 \lambda p \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{2}}\right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right) \tag{19}$$

for some constants $C > 0$ and $\bar{C} > 1$, which do not depend either on λ, p, $|\pi|$, $\bar{n}$ or α. Thus, for a sufficiently small time step $|\pi|$ with respect to λ and p, the global error is such that

$$\left[\mathscr{E}^p + \mathscr{E}^\pi(Y^p)\right]^* = \mathscr{O}\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}\right), \quad \forall \alpha \in \left(0, \tfrac{1}{2}\right).$$

Remark 3 (Global convergence rate). At fixed time step $|\pi|$, the convergence rate strongly deteriorates as λ or p increases, see (19). The numerical method is more sensitive to p than to λ according to (18) and (19). (18) constitutes a necessary condition for the convergence of the backward discrete-time scheme. In practice, the penalization parameter will need to be chosen relatively small and the time step $|\pi|$ very small to avoid multiple jump times on each time step (otherwise, this introduces a bias).

4 Application to Swing Options Valuation

In this section, our method is applied for the valuation of Swing options in the Black and Scholes framework. This constitutes a multiple optimal stopping time problem which can be reformulated as a degenerate three-dimensional impulse control problem. We have been able to achieve convergence for a small maximal number of exercises rights ($n_{\max} \leq 2$) due to the slow computational speed of our method.

4.1 Swing Options Valuation as an Impulse Control Problem

We consider a (normalized) Swing option: the holder of the option is given a maximal number of exercise rights, say $n_{\max} \geq 1$, and has the opportunity to sell whenever he wants over a time period $[0,T]$ an underlying asset against a fixed strike price.

For some fixed strike price K, we shall denote by $\phi(s) = (K-s)^+$ the reward function corresponding to the profit made at each exercise date and by S the underlying asset spot price. We concentrate here on the risk-neutral Black and Scholes framework in which $r > 0$ is a constant interest rate and the spot price process is defined by

$$S_t = S_0 \exp\left\{(r - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}, \quad \forall t \geq 0, \tag{20}$$

where $\sigma > 0$ denotes the volatility coefficient and S_0 an initial price value.

A delay $\delta > 0$ between two consecutive exercise dates is introduced. Indeed, without any delay, the optimal strategy would consist in $n_{\max}$ simultaneous exercise at an unique optimal date (so that this option is equivalent to $n_{\max}$ identical American options). The value of such an option at time 0 can be written as the solution to the following multiple optimal stopping time problem

$$\bar{v}^{(n_{\max})}(s) = \sup_{u=(\tau_k)_{k\geq 1} \in \mathscr{U}^{\delta,n_{\max}}_{(0,T]}} \mathbb{E}\left[\sum_{k\geq 1} e^{-r\tau_k}\phi(S_{\tau_k})\right] \tag{21}$$

in which a strategy $u = (\tau_k)_{k\geq 1}$ is said to be admissible and belongs to $\mathscr{U}^{\delta}_{(0,T]}$ if and only if it is a vector of $\mathbb{F}^W$-stopping times valued in $(0,T]$ with maximal lenght $n_{\max}$

which satisfies the constraint on delay, that is

$$\forall k \geq 1, \quad \tau_{k+1} - \tau_k \geq \delta.$$

As a multiple optimal stopping time problem, this problem can be formulated as a particular (and strongly degenerate) impulse control problem. An impulse control corresponds to a sequence of exercise dates and the intervention gain is written as the payoff function ϕ multiplied by an indicator function which allows to satisfy both the constraint on the number of exercise rights and the constraint on delay between exercise dates. Namely:

$$v(s) = \sup_{u=(\tau_k)_{k\geq 1}} \mathbb{E}\left[\sum_{\substack{k\geq 1\\ \tau_k \leq T}} e^{-r\tau_k}\phi(S_{\tau_k})\mathbb{1}_{\left\{\left(\Theta^{k-1}_{\tau_k^-}\geq\delta\right)\cap\left(Q^u_{\tau_k^-}<n_{\max}\right)\right\}}\right], \tag{22}$$

where $u = (\tau_k)_{k\geq 1}$ is the sequence of $\mathbb{F}^W$-stopping times valued in $(0,T]$ and two additional state variables which both are controlled and discontinuous (càdlàg) processes are introduced:

- Q^u counts the number of exercise rights used before considered time
$$Q^u_0 = 0, \quad Q^u_t = \sharp\{k \geq 1, \tau_k \leq t\}, \quad \forall t \geq 0.$$
- $\Theta^u_t := \Theta^k_t = \inf\{t - \tau_k, \tau_k \leq t\}$ corresponds to the delay between t and last exercise date
$$\Theta^k_t = t - \tau_k, \forall \tau_k \leq t < \tau_{k+1}, \quad \Theta^k_{\tau_{k+1}} = 0, \quad \forall k \geq 0,$$
where by convention $\Theta^u_0 = \Theta^0_0 = 0$.

Obviously, the problem (21) is equivalent to the impulse control problem (22):

$$\forall s \in \mathbb{R}, \quad \bar{v}^{(n_{\max})}(s) = v(s).$$

Penalized BSDE Associated to Swing Option Pricing Problem

Let us introduce the uncontrolled variable (Q,Θ) defined by

$$\begin{cases} Q_t &= N_t, & \forall t \geq 0,\\ \Theta_t &= t - \int_0^t \Theta_{s^-} dN_s, \end{cases} \tag{23}$$

where we recall that N is a Poisson process with intensity $\lambda > 0$. For a penalization coefficient $p > 0$, the penalized BSDE with jumps associated to problem (22) is

$$Y_t^p = \kappa(S_T, Q_{T^-}, \Theta_{T^-}) - \int_t^T rY_u^p du + p\int_t^T (V_u^p + \kappa(S_u, Q_{u^-}, \Theta_{u^-}))^+ \lambda du \\ - \int_t^T Z_u^p dW_u - \int_t^T V_u^p dN_u, \quad \forall 0 \le t \le T \tag{24}$$

in which the intervention gain is defined by

$$\kappa(s,q,\theta) := \phi(s)\mathbb{1}_{\{(\theta \ge \delta)\cap(q \le n_{\max}-1)\}}, \quad \forall (s,q,\theta) \in \mathbb{R}\times\mathbb{N}\times\mathbb{R}^+. \tag{25}$$

4.2 Numerical Valuation Algorithm

The three-dimensional process (S,Q,Θ) can be exactly computed on the time grid π. We shall denote by $(S^\pi, Q^\pi, \Theta^\pi)$ its version on π. In particular, the pure jump processes Q and Θ can be computed without approximation error, recall (23), from a simulated trajectory of a Poisson process with intensity λ, see for example [11].

As the driver of the penalized BSDE with jumps (24) does not depend on Z^p, it is sufficient to compute $(Y^{p,\pi}, V^{p,\pi})$ on π, which can be made by the backward recursive scheme:

$$\begin{cases} Y_{t_N}^{p,\pi} = \kappa(S_{t_N}^\pi, Q_{t_N}^\pi, \Theta_{t_N}^\pi) \\ \forall t_n \in \pi, t_n < T: \\ \qquad V_{t_n}^{p,\pi} = \frac{1}{\lambda\Delta t_{n+1}} \mathbb{E}\left[Y_{t_{n+1}}^{p,\pi} \Delta\tilde{N}_{t_{n+1}} | \mathscr{F}_{t_n}\right] \\ \qquad Y_{t_n}^{p,\pi} = \frac{1}{1+r\Delta t_{n+1}} \left(\mathbb{E}\left[Y_{t_{n+1}}^{p,\pi} | \mathscr{F}_{t_n}\right]\right. \\ \qquad\qquad\qquad \left. + \left[p\left(V_{t_n}^{p,\pi} + \kappa(S_{t_n}^\pi, Q_{t_n}^\pi, \Theta_{t_n}^\pi)\right)^+ - V_{t_n}^{p,\pi}\right] \lambda\Delta t_{n+1}\right) \end{cases} \tag{26}$$

and in our setting:

$$\mathbb{E}\left[\cdot|\mathscr{F}_{t_n}\right] = \mathbb{E}\left[\cdot|(S_{t_n}^\pi, Q_{t_n}^\pi, \Theta_{t_n}^\pi)\right] \ . \tag{27}$$

Monte Carlo-based Resolution

We compute estimators of the conditional expectations (27) by a classical least squares Monte Carlo technique. We use least squares regressions on adaptative local basis functions, see [6]. Since such an approach is only relevant for real-valued variables (the regression basis functions have compact support), it cannot handle the integer-valued variable Q^π.

We thus deal with this variable explicitly. Namely, we simulate $M \ge 1$ i.i.d. paths of $(S^\pi, Q^\pi, \Theta^\pi)$

$$(S^{\pi,(m)}, Q^{\pi,(m)}, \Theta^{\pi,(m)}), \quad \forall m \le M,$$

and at each time step $t_n < T$ of the backward recursion, the set of M Monte Carlo samples is separated in $n_{\max}+1$ sub-sets corresponding to the samples on which $Q^{\pi}_{t_n} = 0,1,\dots,n_{\max-1}$ and $Q^{\pi}_{t_n} \geq n_{\max}$. Then, we just need to estimate $n_{\max}$ conditional expectation operators, namely

$$\mathbb{E}\left[\cdot|(S^{\pi}_{t_n}, Q^{\pi}_{t_n} = q, \Theta^{\pi}_{t_n})\right], \quad \forall q \in \{0,1,\dots,n_{\max}-1\}$$

by using the corresponding Monte Carlo samples, since $(Y^{p,\pi}_{t_n}, V^{p,\pi}_{t_n}) = (0,0)$ when $Q^{\pi}_{t_n} \geq n_{\max}$, see Remark 4.

Remark 4. Let us show that $Q^{\pi}_{t_n} \geq n_{\max}$ implies $Y^{p,\pi}_{t_n} = V^{p,\pi}_{t_n} = 0$. We have

$$\begin{aligned}
&\mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}\Delta\tilde{N}_{t_{n+1}}|(S^{\pi}_{t_n}, Q^{\pi}_{t_n} \geq n_{\max}, \Theta^{\pi}_{t_n})\right] \\
&= \mathbb{E}\Big[\underbrace{\mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}|Q^{\pi}_{t_n} \geq n_{\max}\right]}_{=0}\Delta\tilde{N}_{t_{n+1}}|(S^{\pi}_{t_n}, Q^{\pi}_{t_n} \geq n_{\max}, \Theta^{\pi}_{t_n})\Big] \;=\; 0,
\end{aligned}$$

by definition of κ in (25) and (26). In the same way

$$\mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}|(S^{\pi}_{t_n}, Q^{\pi}_{t_n} \geq n_{\max}, \Theta^{\pi}_{t_n})\right] \;=\; 0$$

which allows to conlude.

The Monte Carlo-based algorithm is then the following:

I. Initialization:

$$Y^{p,\pi,(m)}_{t_N} = \phi(S^{\pi,(m)}_{t_N})\mathbb{1}_{\left\{\left(\Theta^{\pi,(m)}_{t_N} \geq \delta\right)\cap\left(Q^{\pi,(m)}_{t_N} \leq n_{\max}-1\right)\right\}}, \quad \forall m \leq M.$$

II. Computation backward in time of $(V^{p,\pi,(m)}, Y^{p,\pi,(m)})$ on each sample $m \leq M$. For $n = N-1,\dots,0$, set

$$\begin{aligned}
\mathscr{M}^{q}_{t_n} &:= \left\{m = 1,\dots,M : Q^{\pi,(m)}_{t_n} = q\right\}, \quad \forall q \leq n_{\max}-1, \\
\mathscr{M}^{n_{\max}}_{t_n} &:= \left\{m = 1,\dots,M : Q^{\pi,(m)}_{t_n} \geq n_{\max}\right\}.
\end{aligned}$$

Then:

1. For any $m \in \mathscr{M}^{n_{\max}}_{t_n}$,
$$V^{p,\pi,(m)}_{t_n} = Y^{p,\pi,(m)}_{t_n} = 0.$$
2. Set $q := n_{\max}-1$.
3. If $q \geq 0$, for any $m \in \mathscr{M}^{q}_{t_n}$, the conditional expectations estimators

$$\begin{aligned}
\varepsilon^{V,q,(m)}_{t_n} &\approx \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}\Delta\tilde{N}_{t_{n+1}}|\;\left(S^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}\right)\right] \\
\varepsilon^{Y,q,(m)}_{t_n} &\approx \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}|\;\left(S^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}\right)\right]
\end{aligned}$$

are approximated by least squares regression of

$$\left(Y^{p,\pi,(m)}_{t_{n+1}} \Delta\tilde{N}^{(m)}_{t_{n+1}}\right)_{m\in\mathscr{M}^q_{t_n}} \text{ and } \left(Y^{p,\pi,(m)}_{t_{n+1}}\right)_{m\in\mathscr{M}^q_{t_n}}$$

respectively on local basis functions (see the precise description of the procedure in [6]). We shall denote by $b^{q,S}_{t_n}$ and $b^{q,\Theta}_{t_n}$ the numbers of basis functions used in each direction of the state (S^π, Θ^π). Then,

$$\begin{cases} V^{p,\pi,(m)}_{t_n} = \frac{1}{\lambda\Delta t_{n+1}}\ \varepsilon^{V,q,(m)}_{t_n} \\ Y^{p,\pi,(m)}_{t_n} = \frac{1}{1+r\Delta t_{n+1}} \left(\varepsilon^{Y,q,(m)}_{t_n} + \left[p\left(V^{p,\pi,(m)}_{t_n} + \phi(S^{\pi,(m)}_{t_n})\mathbb{1}_{\left\{\Theta^{\pi,(m)}_{t_n}\geq\delta\right\}}\right)^+ \right.\right. \\ \qquad\qquad\qquad\qquad \left.\left. - V^{p,\pi,(m)}_{t_n}\right]\lambda\Delta t_{n+1}\right). \end{cases}$$

4. $q := q-1$ and go to 3.

III. At time t_0 ($\mathscr{M}^0_{t_0} = \{1,\ldots,M\}$ and $\mathscr{M}^q_{t_0} = \emptyset, \forall q \geq 1$) the Swing option price estimator is given by $Y^{p,\pi}_{t_0}$ such that

$$\begin{cases} V^{p,\pi}_{t_0} = \frac{1}{\lambda\Delta t_1}\frac{1}{M}\sum_{m=1}^M \left(Y^{p,\pi,(m)}_{t_1}\Delta\tilde{N}^{(m)}_{t_1}\right) \\ Y^{p,\pi}_{t_0} = \frac{1}{1+r\Delta t_1}\left(\frac{1}{M}\sum_{m=1}^M Y^{p,\pi,(m)}_{t_1} + \left[p(V^{p,\pi}_{t_0})^+ - V^{p,\pi}_{t_0}\right]\lambda\Delta t_1\right). \end{cases}$$

Let us highlight some features of the above-presented Monte Carlo procedure. At each backward induction date $t_n < T$, we have to estimate in worst cases $2\times n_{\max}$ conditional expectations, performed on each subset $\mathscr{M}^q_{t_n}, q \leq n_{\max}-1$. When $n_{\max}$ increases, much more Monte Carlo samples are needed as each least squares regression requires a sufficient number of samples.

In addition, the numbers of local basis functions $(b^{q,S}_{t_n}, b^{q,\Theta}_{t_n})$ have to be adapted to the number of Monte Carlo samples used for the least squares regression, namely $\mathrm{card}(\mathscr{M}^q_{t_n})$. We thus introduce a dynamic choice: $(b^{q,S}_{t_n}, b^{q,\Theta}_{t_n})$ are fixed proportionally to $\mathrm{card}(\mathscr{M}^q_{t_n})$ for any $q \leq n_{\max}-1$ and $t_n < T$. As an example, in the case of American options below ($n_{\max} = 1$), $b^S_{t_n}$ varies from 2 to 10 (the additional variable Θ disappears) and in the case of Swing options with $n_{\max} = 2$ exercise rights, $b^S_{t_n}$ varies from 2 to 20 and $b^{q,\Theta}_{t_n}$ from 2 to 10.

Remark 5 (Statistical error of our method). A control of the statistical error introduced by the least squares Monte Carlo approach is provided in Gobet et al. [18] (see [23] for further details). By extension, this applies to BSDE with jumps (see [15]) to control the error on the jump component $V^{p,\pi}$ and thus ensures that the least squares Monte Carlo error tends to 0 as the number of samples M and the number of basis functions b tends to $+\infty$.

A Benchmark Method Based on Iteration

The classical method to value such a Swing option, recall formulation (21), is based on an iteration over the number of exercise rights, see for example [9]. The dynamic programming principle provides a direct link between the solution $v^{(j)}$ to the same problem as (21) but with at most $j \leq n_{\max}$ exercise rights and the solution $v^{(j-1)}$ with at most $(j-1)$ exercise rights.

The value of the Swing option with 0 exercise right is obviously zero $v^{(0)} = 0$. Then, we compute the sequence of values of Swing options with j exercise rights $v^{(j)}$, $j = 1, \dots, n_{\max}$ according the backward recursion scheme:

$$\begin{cases} v^{(j)}(t_N, s) = \phi(s) \\ \forall t_n \in \pi, T - \delta < t_n < T : \\ \qquad v^{(j)}(t_n, s) = \max\left\{ \phi(s) \; ; \; e^{-r\Delta t_{n+1}} \mathbb{E}^{(t_n,s)}\left[v^{(j)}(t_{n+1}, S^{\pi}_{t_{n+1}}) \right] \right\} \\ \forall t_n \in \pi, t_n \leq T - \delta : \\ \qquad v^{(j)}(t_n, s) = \max\left\{ \phi(s) + e^{-r\delta} \mathbb{E}^{(t_n,s)}\left[v^{(j-1)}(t_n + \delta, S^{\pi}_{t_n+\delta}) \right] ; \right. \\ \qquad\qquad\qquad \left. e^{-r\Delta t_{n+1}} \mathbb{E}^{(t_n,s)}\left[v^{(j)}(t_{n+1}, S^{\pi}_{t_{n+1}}) \right] \right\} \end{cases}$$

where $\mathbb{E}^{(t_n,s)}[\cdot] := \mathbb{E}\left[\cdot | S^{\pi}_{t_n} = s\right]$. We use the same least squares Monte Carlo regression-based method for approximating the conditional expectations operators as above.

4.3 Pricing Results

We consider put options with maturity $T = 1$ year and a strike price $K = 100$. The Black and Scholes parameters, see (20), are $r = 0.05$, $\sigma = 0.3$ and $S_0 = 100$.

4.3.1 Special Case of American Options: $n_{\max} = 1$

In the single-exercise case, the additional variable Θ disappears (there is no delay constraint). This helps simplify the Monte Carlo procedure described in Paragraph 4.2. In particular, the algorithm implies only two sequences of samples subsets, whether one jump of N occurs before T on the considered path or not: that is $((\mathscr{M}^1)_{t_n})_{n \leq N-1}$ and $((\mathscr{M}^0)_{t_n})_{n \leq N-1}$.

In our numerical experiments, we find out that increasing too much λ makes the variance of the Monte Carlo procedure explode. It would be necessary to increase the number of Monte Carlo samples, which leads to prohibitive computational times (each pricing result presented below was obtained after a computation between 6 and 8 hours). For the same reason (exploding behavior of the penalized

BSDE driver), we restrict our numerical experiments to penalization parameters ≤ 5.

The benchmark price for the American put option is 9.88 (by a binomial approach or classical Monte Carlo). We report in Table 1 the price given by our method when varying λ and the number of time steps N for a penalization parameter equal to 5. We used 20 million of Monte Carlo paths.

Table 1 Approximate prices of an American option with $p = 5$

λ \ N	20	40	80	160	320
3	9.89	9.92	9.95	9.94	9.83
4	9.92	9.96	9.99	9.97	9.83
5	9.95	9.99	10.02	9.98	9.76

In all the experiments that we performed in this simple case, we numerically observed that the limiting prices of our method (with respect to N) are below the benchmark value: this is due to penalization.

4.3.2 Swing Options with $n_{\max} = 2$

We consider time delays $\delta = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$.

The benchmark prices for the Swing put option with 2 exercise rights are 19.27, 18.77 and 18.21 respectively (computed with the method described in Paragraph 4.2, $N = 200$ time steps and $M = 5$ million of Monte Carlo paths). We report in Figures 1, 2 and 3 the corresponding approximate prices when varying λ and N for a penalization parameter equal to 5 and 10 (we used 40 million of Monte Carlo paths and $N = 20, 40, 80, 160, 320, 640$).

For each considered value of λ, we retrieve a convergence in the number of time steps N of our method. As $p = 5$, approximate prices are converged from $N = 160$, so that we restrict ourselves to $N \leq 320$ time steps as $p = 10$. The limiting values are still below the benchmark but accurate option prices (relative error less than 1%) are obtained with a penalization coefficient p equal to 10 and $N = 160$. See also Table 2 in which the (signed) relative error to the benchmark is given in brackets. Besides, we observe a monotone convergence in λ of our approximate method.

As shown on the Figures 1, 2 and 3 and Table 2, λ needs to be sufficiently large. This is in agreement with the estimates of propositions 1 and 2. For too small λ, the option's values given by the method are not converged. It can been interpreted as follows: the value process does not jump often enough and the algorithm has difficulties to capture the whole set of possible exercice dates, see also in [15] p. 146. However, if λ is too large, the variance of the Monte Carlo method increases.

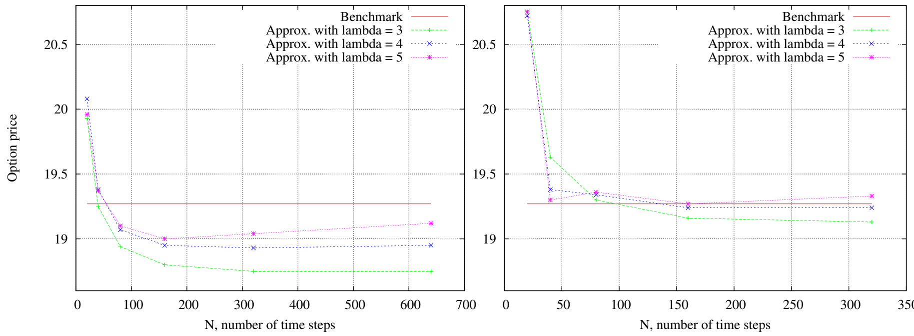

Fig. 1 Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{1}{10}$, with $p = 5$ (left) and $p = 10$ (right)

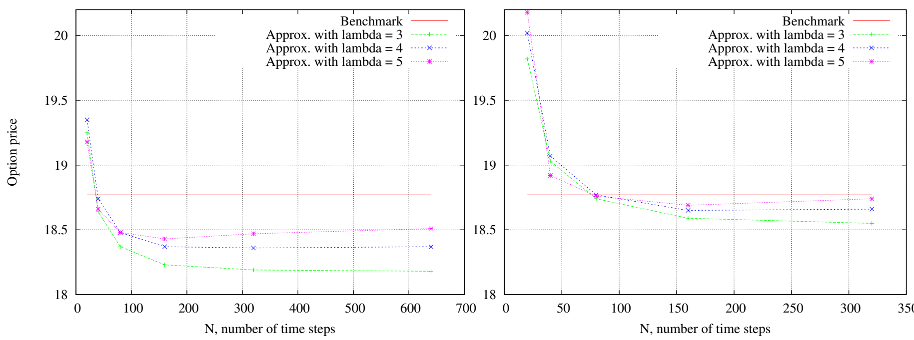

Fig. 2 Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{2}{10}$, with $p = 5$ (left) and $p = 10$ (right)

Table 2 Prices of a Swing option with 2 exercise rights (limiting values with $N = 160$)

	p \ λ	3	4	5
$\delta = \frac{1}{10}$	5	18.80 (-2.44%)	18.95 (-1.66%)	19.00 (-1.40%)
	10	19.16 (-0.57%)	19.24 (-0.16%)	19.27 (0.00%)
$\delta = \frac{2}{10}$	5	18.23 (-2.88%)	18.37 (-2.13%)	18.43 (-1.81%)
	10	18.59 (-0.96%)	18.65 (-0.64%)	18.69 (-0.43%)
$\delta = \frac{3}{10}$	5	17.62 (-2.97%)	17.76 (-2.20%)	17.81 (-1.93%)
	10	17.99 (-0.94%)	18.04 (-0.66%)	18.05 (-0.61%)

We should point out that fine-tuning the parameters of the algorithm is difficult. As already mentioned, since the number of Monte Carlo paths is different in each set of sample paths $\mathcal{M}_{t_n}^q, q = 0, 1, 2$, the number of basis functions used for the least

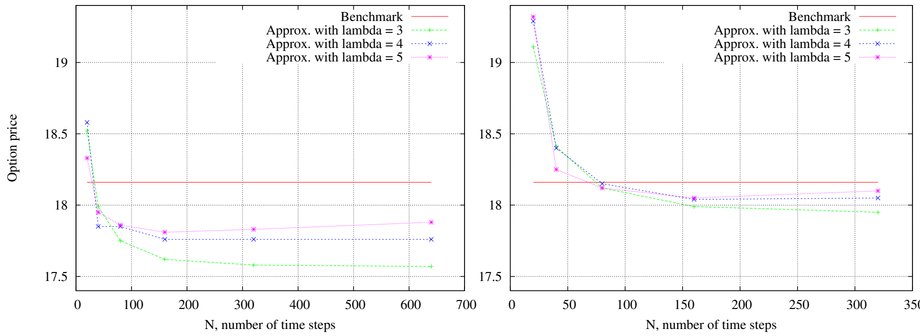

Fig. 3 Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{3}{10}$, with $p = 5$ (left) and $p = 10$ (right)

squares regressions has to be dynamically adapted. And when increasing much more the jump intensity λ, more Monte Carlo samples would be necessary.

For such a Swing option, the running time is much longer because the conditional expectations are computed by regression with respect to the bidimensional state variable (S^π, Θ^π). The computation of one option price takes at least 15 hours in above cases (when $N \geq 80$). In comparison, the benchmark method takes less than 5 minutes. Besides, the complexity of our method increases with $n_{\max}$, leading to untractable computational times for bigger values of $n_{\max}$, see Remark 6.

On this particular case of Swing options valuation, it seems that our method is less competitive than the classical approach. This is without any doubt due to the strong degeneracy of such a problem in our impulse control context: the valuation problem is 3-dimensional and involves an additional integer-valued state variable Q representing the number of exercise rights used at any considered time.

However, our method works and the numerical results that we obtain are consistent with the theoretical convergence rate given in Theorem 1. One can expect that our method would work better on less degenerate problems.

Remark 6 (Dealing with more exercise rights). The computational time of our method intuitively increases *linearly* with the number of exercise rights $n_{\max}$. Indeed, at each time step of the backward induction procedure, the number of conditional expectation estimations is proportional to $n_{\max}$. Besides, when multiplying by 2 the number of exercise rights, it would require, at least, a double number of Monte Carlo samples for a same accuracy of the computation of conditional expectation estimators.

Let us mention that the computational time of the benchmark method using iteration increases linearly as function of the the maximal number of exercise rights as well.

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