

Variance optimal hedging with application to Electricity markets

Xavier Warin^{*†}

November 10, 2017

Abstract

In Electricity markets, illiquidity, transaction costs and market price characteristics prevent managers to replicate exactly contracts. A residual risk is always present and the hedging strategy depends on a risk criterion chosen. We present an algorithm to hedge a position for a mean variance criterion taking into account the transaction cost and the small depth of the market. We show its effectiveness on a typical problem coming from the field of electricity markets.

Keywords: Monte-Carlo methods, mean variance, energy, finance

MSC2010: Primary 65C05, 60J60; Secondary 60J85, 60H35.

1 Introduction

Since the deregulation of the energy market in the 1990, spot and future contract on electricity are available in many countries. Because electricity cannot be stored easily and because balance between production and consumption has to be checked at every moment, electricity prices exhibit spikes when there is a high demand or a shortage in production. These high variations in the price lead to fat tails of the distribution of the log return of the price. Besides when the production and demand return to normal levels, the price goes back to an average value : this is the mean reverting effect.

In order to model this price behavior mainly two kinds of models have emerged :

- the HJM style forward curves models permits to model directly the dynamic of the forward curve. Its first use for commodity was due to [19] for crude oil. The dynamic of the curve at date t for a delivery at date T is given under the risk neutral probability by

$$\frac{dF}{F}(t, T) = \sum_{i=1}^N \sigma_i(t, T) dW_t^i. \quad (1.1)$$

^{*}EDF R&D & FiME, Laboratoire de Finance des Marchés de l'Energie

[†]warin@edf.fr

In order to get a mean reverting effect on the spot and short term future values, a two factors model is often used :

$$\frac{dF}{F}(t, T) = e^{-a(T-t)}\sigma_S dW_t^S + \sigma_L dW_t^L, \quad (1.2)$$

with W_t^S and W_t^L two Brownian motions potentially correlated.

Due to the use of Brownian motion to model uncertainty and the availability of different products to hedge a position, this market is complete.

However the model (1.1) doesn't permit to get log returns of the spot prices coherent to what is seen on the markets. To get more realistic returns it is possible to use some Levy models such as the Normal Inverse Gaussian model to model the prices (see [5]). But then the market becomes incomplete and a selection of a risk neutral probability has to be achieved either by the use of an optimization criterion or by a parametrization of the change of probability by using for example an Esscher transform and a calibration based on options available on the market [4].

- The second way to model prices (at least spot price) is to use structural models : starting with a consumption model and potentially adding some model of the fuels involved in the generation of electricity, these models mimic the price formation mechanism. The first one proposed by [2] only involved the consumption level. A more complex structural model was proposed by [1] and a survey on the methods can be found in [9]. All these models involve the demand curve which is no hedgeable and thus lead to incomplete markets.

All previous model except the Gaussian HJM (1.1) lead to incomplete market. However even in the case of a Gaussian model there are many sources of friction and incompleteness on energy market :

- The first is the liquidity of hedging products available on the market. Big change in position leads to change in prices and then the hypothesis of an exogenous price model is not valid anymore. In order to limit their impacts on price, risk managers spread in time their selling/buying orders. A realistic hedging strategy has to take these constraints into account.
- Due to the spread bid ask which is already a source of friction and due to the inaccuracy of the models used, hedging is only achieved once a week or twice a month so that the hypothesis of a continuous hedge is far from being satisfied,
- At last on Electricity market, some contracts are not only dependent on the prices. For example a retailer has to provide the energy to fit a load curve which is uncertain mainly due to thermosensibility and business activity and a perfect hedge of such contracts is impossible.

The literature on the effect of a discrete hedging has been theoretically studied in complete markets in [33], [6], [18], and some asymptotic "almost" optimal strategies obtained by selecting the hedging dates are defined in [14], [22]. In the case of incomplete markets with some jumps some results about the hedge error due to discrete hedging can be found in [31],

and an extension Fukasawa's work can be found in [27].

When dealing with discrete time hedging and transaction cost, the Lelang proxy ([23]), the first proposed, is generally used for Gaussian models in the case of proportional transaction costs. One theoretical result obtained gives that for proportional transaction costs decreasing with the number of interventions n proportionally to $n^{-\alpha}$ with $\alpha \in]0, \frac{1}{2}]$ the hedging error goes to zero in probability when a call is valued and hedged using the Black Scholes formula with a modified volatility.

Following this first work, a huge literature on transaction cost has developed. For example, the case $\alpha = 0$ corresponding to the realistic case where the transaction cost are independent on the frequency to the hedge has been dealt in [20], [26], [12], [30].

The case of a limited availability of hedging products has not been dealt theoretically nor numerically so far in the literature to our knowledge. Most of the research currently focuses on developing some price impact features to model the liquidation of a position (see for example [15]) instead of imposing some depth limits.

In order to be get realistic hedging strategies and because of all the sources of incompleteness, a risk criterion has to be used to define the optimal hedging strategy of the contingent claim giving a payoff at maturity date T . In this article we will use a mean variance strategy to define this optimal policy taking into account the fact that hedging is achieved at discrete dates, that transaction costs are present and that the orders are limited in volume at each date.

In a first part, in a general setting using the previous work of [28], [25] and [7], we show that the problem admits a solution in a general case.

Then we develop some dynamic programming algorithms that permit to efficiently calculate the optimal strategy associated to the mean variance problem. Mean variance algorithm without friction and based on tree resolution have been developed by [10] but our algorithm is more closely related to the framework proposed by [11]: using Monte Carlo methods and the methodology first developed in [24] to track optimal cash flows strategies along trajectories we are able to provide a very effective algorithm.

In a third part, we focus on the problem of an energy retailer that needs to hedge an open position corresponding to a stochastic load curve by trading future contracts. Using a Gaussian one factor HJM model for the future curve and and Ornstein Ulhenbeck model for the load curve, we will show that on Electricity market, taking into account the discrete hedge and the limited orders has a high impact on the optimal strategy.

2 Mean variance hedging in a general framework

In this section we suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space adapted to a filtration $F = (\mathcal{F}_t)_{t \in [0, T]}$. We define a set of trading dates $\mathcal{T} = \{t_0 = 0, t_1, \dots, t_{N-1}, t_N = T\}$ and we suppose that we are given an asset used as an hedging product $(S_t)_{t_0, t_N}$ which is almost surely positive, square integrable so that $\mathbb{E}[S_t^2] < \infty$ and adapted so that S_t is \mathcal{F}_t -measurable for $t = t_0, \dots, t_N$.

At last we suppose that the risk free rate is zero so that a bond has always a value of 1.

We suppose that we are given a contingent claim $H \in \mathcal{L}^2(P)$ which is supposed to be a \mathcal{F}_T -measurable random variable. In the case of a European call option on an asset S_t with strike K and maturity T , $H(\omega) = (S_T(\omega) - K)^+$.

In this paper we are only interested in self financing strategies with limited orders. Extending [25], [7] definition,

Definition 2.1. A (\bar{m}, \bar{l}) self-financing strategy $\mathcal{V} = (\mathcal{V}_{t_i})_{i=0, \dots, N-1}$ is a pair of adapted process $(m_{t_i}, l_{t_i})_{i=0, \dots, N-1}$ defined for $(\bar{m}, \bar{l}) \in \mathbb{R}^{+*} \times \mathbb{R}^{+*}$ such that:

- $0 \leq m_{t_i} \leq \bar{m}, \quad 0 \leq l_{t_i} \leq \bar{l} \quad P.a.s. \quad \forall i = 0, \dots, N-1,$
- $m_{t_i} l_{t_i} = 0 \quad P.a.s. \quad \forall i = 0, \dots, N-1.$

In this definition m_t corresponds to the number of shares sold at date t , and l_t the number of share bought at this date.

Remark 2.2. The strategies defined in [25], [7] doesn't impose that $m_t l_t = 0$ so that we are not assure that we don't have a buy and sell strategy at the same date.

We note $\Theta^{(\bar{m}, \bar{l})}$ the set of (\bar{m}, \bar{l}) self-financing strategy and with obvious notations $\nu = (m, l)$ for $\nu \in \Theta^{(\bar{m}, \bar{l})}$.

We consider a model of proportional cost, so that an investor buying a share at date t will pay $(1 + \lambda)S_t$ and an investor selling this share will only receive $(1 - \lambda)S_t$. Assuming no transaction cost on the last date T , the terminal wealth of an investor with initial wealth x is given by :

$$x - \sum_{i=0}^{N-1} (1 + \lambda)l_{t_i}S_{t_i} + \sum_{i=0}^{N-1} (1 - \lambda)m_{t_i}S_{t_i} + \sum_{i=0}^{N-1} l_{t_i}S_{t_N} - \sum_{i=0}^{N-1} m_{t_i}S_{t_N}. \quad (2.1)$$

As in [25] [7], we define the risk minimal strategy minimizing the \mathcal{L}^2 risk of the hedge portfolio :

Definition 2.3. A (\bar{m}, \bar{l}) self-financing strategy $\hat{\mathcal{V}} = (\hat{m}, \hat{l})$ is global risk minimizing for the contingent claim H and the initial capital x if :

$$\begin{aligned} \hat{\mathcal{V}} = \arg \min_{\nu=(m,l) \in \Theta^{(\bar{m}, \bar{l})}} \mathbb{E} & \left[\left(H - x + \sum_{i=0}^{N-1} (1 + \lambda)l_{t_i}S_{t_i} - \right. \right. \\ & \left. \left. \sum_{i=0}^{N-1} (1 - \lambda)m_{t_i}S_{t_i} - \sum_{i=0}^{N-1} l_{t_i}S_{t_N} + \sum_{i=0}^{N-1} m_{t_i}S_{t_N} \right)^2 \right]. \end{aligned} \quad (2.2)$$

In order to show the existence of solution to problem (2.2), we need to make some of the assumptions on the prices as used in [25], [7] :

Assumption 2.4. The price process $S = (S_{t_i})_{i=0, \dots, N}$ is such that a constant $K_1 > 0$ satisfies:

$$\mathbb{E} \left[\frac{S_{t_i}^2}{S_{t_{i-1}}^2} \middle| \mathcal{F}_{t_{i-1}} \right] \leq K_1, \quad P.a.s., \quad \forall i = 1, \dots, N.$$

Assumption 2.5. The price process $S = (S_{t_i})_{i=0,\dots,N}$ is such that a constant $K_2 > 0$ satisfies:

$$\mathbb{E}\left[\frac{S_{t_{i-1}}^2}{S_{t_i}^2} \mid \mathcal{F}_{t_{i-1}}\right] \leq K_2, \quad P.a.s., \quad \forall i = 1, \dots, N.$$

Assumption 2.6. The price process $S = (S_{t_i})_{i=0,\dots,N}$ is such that a constant $\delta \in (0, 1)$ satisfies :

$$(\mathbb{E}[S_{t_i} \mid \mathcal{F}_{t_{i-1}}])^2 \leq \delta \mathbb{E}[S_{t_i}^2 \mid \mathcal{F}_{t_{i-1}}], \quad P.a.s., \quad \forall i = 1, \dots, N.$$

We introduce a gain function :

Definition 2.7. For $\mathcal{V} \in \Theta^{(\bar{m}, \bar{l})}$, we define the gain functional $G_T : \Theta^{(\bar{m}, \bar{l})} \longrightarrow \mathcal{L}^2$ by

$$G_T(\mathcal{V}) := - \sum_{i=0}^{N-1} (1 + \lambda) l_{t_i} S_{t_i} + \sum_{i=0}^{N-1} l_{t_i} S_T + \sum_{i=0}^{N-1} (1 - \lambda) m_{t_i} S_{t_i} - \sum_{i=0}^{N-1} m_{t_i} S_T \quad (2.3)$$

In order to show that our problem is well defined, we want to show that $G_T(\Theta^{(\bar{m}, \bar{l})})$ is closed in \mathcal{L}^2 . Using similar idea as in [25], [7], we have to introduce another functional $\tilde{G}_T(\mathcal{V}) : \Theta^{(\bar{m}, \bar{l})} \longrightarrow \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$ defined by:

$$\tilde{G}_T(\mathcal{V}) := \begin{pmatrix} - \sum_{i=0}^{N-1} (1 + \lambda) l_{t_i} S_{t_i} \\ \sum_{i=0}^{N-1} (1 - \lambda) m_{t_i} S_{t_i} \\ \sum_{i=0}^{N-1} (l_{t_i} - m_{t_i}) S_T \end{pmatrix} \quad (2.4)$$

Proposition 2.8. Under assumptions 2.4, 2.5, 2.6, $\tilde{G}_T(\Theta^{(\bar{m}, \bar{l})})$ is a closed bounded convex set of $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$.

Proof. The convexity of the set is obvious due to the linearity of the \tilde{G}_T operator. Because of the boundedness of (l_{t_i}, m_{t_i}) , $i = 0, N - 1$ and the $(S_{t_i})_{i=0,N}$ square integrability, we have that $\tilde{G}_T(\Theta^{(\bar{m}, \bar{l})})$ is bounded in $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$.

In order to prove the closeness of $\tilde{G}_T(\Theta^{(\bar{m}, \bar{l})})$, we suppose that we have a sequence $(\mathcal{V}^n) = (m_{t_i}^n, l_{t_i}^n)_{i=0, N-1} \in \Theta^{(\bar{m}, \bar{l})}$ such that $\tilde{G}_T(\mathcal{V}^n)$ is converging in $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$.

Using assumption 2.6 and lemma 5.3 in [25], we get that $\forall i = 0, \dots, N - 1$, there exist $l_{t_i}^\infty$, $m_{t_i}^\infty$ \mathcal{F}_{t_i} -adapted such that $l_{t_i}^\infty S_{t_i} \in \mathcal{L}^2$, $m_{t_i}^\infty S_{t_i} \in \mathcal{L}^2$ and

$$\begin{aligned} l_{t_i}^n S_{t_i} &\xrightarrow{\mathcal{L}^2} l_{t_i}^\infty S_{t_i}, \\ m_{t_i}^n S_{t_i} &\xrightarrow{\mathcal{L}^2} m_{t_i}^\infty S_{t_i}, \end{aligned}$$

so that

$$\begin{aligned} - \sum_{i=0}^{N-1} (1 + \lambda) l_{t_i}^n S_{t_i} &\xrightarrow{\mathcal{L}^2} - \sum_{i=0}^{N-1} (1 + \lambda) l_{t_i}^\infty S_{t_i}, \\ \sum_{i=0}^{N-1} (1 - \lambda) m_{t_i}^n S_{t_i} &\xrightarrow{\mathcal{L}^2} \sum_{i=0}^{N-1} (1 - \lambda) m_{t_i}^\infty S_{t_i}. \end{aligned}$$

Using Lemma 5 in [7] based on assumption 2.4, we get that $l_{t_i}^n S_T$ converge weakly to $l_{t_i}^\infty S_T$, and $m_{t_i}^n S_T$ converge weakly to $m_{t_i}^\infty S_T$, for $i = 0, \dots, N-1$ such that

$$\sum_{i=0}^{N-1} (l_{t_i}^n - m_{t_i}^n) S_T \xrightarrow{\text{weak}} \sum_{i=0}^{N-1} (l_{t_i}^\infty - m_{t_i}^\infty) S_T.$$

Using the fact that strong and weak limit coincide in \mathcal{L}^2 , we get that the limit of $\tilde{G}_T(\mathcal{V}^n)$ is $\tilde{G}_T(\mathcal{V}^\infty)$ with $\mathcal{V}^\infty = (m^\infty, l^\infty) \in \mathcal{L}^2 \times \mathcal{L}^2$.

We still have to show that $l_{t_i}^\infty, m_{t_i}^\infty$ respect the constraints: First using assumption 2.4 and the tower property of conditional expectation, we have

$$\mathbb{E}\left[\frac{1}{S_{t_i}^2}\right] \leq \frac{K_2^i}{S_0^2} < \infty, \forall i = 0, \dots, N \quad (2.5)$$

Suppose for example that there exist $\epsilon > 0$ and $i < N$ such that $l_{t_i}^\infty(\omega) > \bar{l} + \epsilon$ for ω in a set $\tilde{\mathcal{A}}$ with a non zero measure $\tilde{\mathcal{A}}$. Then

$$\begin{aligned} (\bar{l} + \epsilon)\tilde{\mathcal{A}} &\leq E[l_{t_i}^\infty 1_{\tilde{\mathcal{A}}}] \\ &= E[(l_{t_i}^\infty - l_{t_i}^p) 1_{\tilde{\mathcal{A}}}] + E[l_{t_i}^p 1_{\tilde{\mathcal{A}}}] \\ &\leq E[(l_{t_i}^\infty S_T - l_{t_i}^p S_T) 1_{\tilde{\mathcal{A}}} \frac{1}{S_T}] + \bar{l}\tilde{\mathcal{A}} \end{aligned}$$

Then using (2.5) we know that $1_{\tilde{\mathcal{A}}} \frac{1}{S_T}$ is in \mathcal{L}^2 so using the weak convergence of $l_{t_i}^p S_T$ towards $l_{t_i}^\infty S_T$ and letting p go to infinity gives the contradiction.

The same type of calculation shows that $l_{t_i}^\infty \geq 0, 0 \leq m_{t_i}^\infty \leq \bar{m}$.

At last it remains to show that $l^\infty m^\infty = 0$. Suppose for example that there exist $\epsilon > 0$ and $i < N$ such that $l_{t_i}^\infty(\omega) m_{t_i}^\infty(\omega) > \epsilon$ for ω in a set $\tilde{\mathcal{A}}$ with a non zero measure $\tilde{\mathcal{A}}$.

Using the fact that $l_{t_i}^n m_{t_i}^n = 0, \frac{m_{t_i}^n 1_{\tilde{\mathcal{A}}}}{S_{t_i}} \in \mathcal{L}^2$ and Cauchy Schwartz:

$$\begin{aligned} \epsilon\tilde{\mathcal{A}} &\leq E[l_{t_i}^\infty m_{t_i}^\infty 1_{\tilde{\mathcal{A}}}] \\ &= \mathbb{E}\left[\left(S_{t_i} m_{t_i}^\infty - S_{t_i} m_{t_i}^n\right) \frac{l_{t_i}^\infty 1_{\tilde{\mathcal{A}}}}{S_i}\right] + \mathbb{E}\left[\left(S_{t_i} l_{t_i}^n - S_{t_i} l_{t_i}^\infty\right) \frac{m_{t_i}^n 1_{\tilde{\mathcal{A}}}}{S_{t_i}}\right] \\ &\leq \mathbb{E}\left[\left(S_{t_i} m_{t_i}^\infty - S_{t_i} m_{t_i}^n\right) \frac{l_{t_i}^\infty 1_{\tilde{\mathcal{A}}}}{S_i}\right] + \|S_{t_i} l_{t_i}^n - S_{t_i} l_{t_i}^\infty\|_{\mathcal{L}^2} \bar{m} \tilde{\mathcal{A}} \frac{K_2^{0.5i}}{S_0} \end{aligned}$$

Letting n going to infinity, and using the strong/weak convergence property we get the contradiction. ■

We are now in position to give the existence and uniqueness theorem:

Theorem 2.9. *Under assumptions 2.4, 2.5, 2.6, the minimization problem 2.2 admits an unique solution.*

Proof. First we show that $G_T(\Theta^{(\bar{m}, \bar{l})})$ is a closed bounded convex set in \mathcal{L}^2 . Convexity and boundedness are obvious. Suppose that we have a sequence $(\mathcal{V}^n) = (m_{t_i}^n, l_{t_i}^n)_{i=0, N-1} \in \Theta^{(\bar{m}, \bar{l})}$

such that $G(\mathcal{V}^n)$ converges in \mathcal{L}^2 . $\tilde{G}_T(\Theta^{(\bar{m}, \bar{l})})$ being a bounded closed set due to proposition 2.8, it is weakly closed and we can extract a sub sequence \mathcal{V}^p such that $\tilde{G}_T(\mathcal{V}^p)$ converge weakly to $\tilde{G}_T(\mathcal{V}^\infty)$ with $\mathcal{V}^\infty \in \Theta^{(\bar{m}, \bar{l})}$. Then by linearity $G_T(\mathcal{V}^p)$ converges weakly to $G_T(\mathcal{V}^\infty)$ and due to the fact that weak and strong limit coincide, we get that $G_T(\mathcal{V}^p)$ converges strongly to $G_T(\mathcal{V}^\infty)$ giving the closeness of $G_T(\Theta^{(\bar{m}, \bar{l})})$.

As a result, the solution of the minimization problem exists, is unique and corresponds to the projection of $H - x$ on $G_T(\Theta^{(\bar{m}, \bar{l})})$ in \mathcal{L}^2 . ■

3 Algorithms for mean variance hedging

In this section we suppose that the process is Markov and that the payoff H is a function of the asset value at maturity only to simplify the presentation for the Monte Carlo method proposed. In a first section we show that we can get a dynamic program method to easily calculate the solution of problem (2.2). In a second part we give the practical algorithm to solve the problem.

3.1 A dynamic programming algorithm

We introduce the global position $\nu = (\nu_i)_{i=0, \dots, N-1}$ with:

$$\nu_i = \sum_{j=0}^i (m_{t_j} - l_{t_j}), \forall i = 0, \dots, N-1.$$

Using the property that $m_{t_i} l_{t_i} = 0$, $\forall i = 0, \dots, N-1$, we get $|\nu_i - \nu_{i-1}| = l_{t_i} + m_{t_i}$ with the convention that $\nu_{-1} = 0$ and

$$G_T(\mathcal{V}) = \hat{G}_T(\nu) = x - \sum_{i=0}^{N-1} \lambda |\Delta \nu_{i-1}| S_{t_i} + \sum_{i=0}^{N-1} \nu_i \Delta S_i,$$

where $\Delta S_i = S_{t_{i+1}} - S_{t_i}$, $\Delta \nu_i = \nu_{i+1} - \nu_i$.

We then introduce $\hat{\Theta}^{(\bar{m}, \bar{l})}$ the set of adapted random variable $(\nu_i)_{i=0, \dots, N-1}$ such that

$$-\bar{m} \leq \nu_i - \nu_{i-1} \leq \bar{l}, \forall i = 1, \dots, N-1.$$

The problem (2.2) can be rewritten as done in [28] finding $\hat{\nu} = (\hat{\nu}_i)_{i=0, \dots, N-1}$ satisfying:

$$\hat{\nu} = \arg \min_{\nu \in \hat{\Theta}^{(\bar{m}, \bar{l})}} \mathbb{E}[(H - x - \hat{G}_T(\nu))^2]. \quad (3.1)$$

We introduce the spaces κ_i , $i = 0, \dots, N$ of the \mathcal{F}_{t_i} -measurable and square integrable random variables. We define for $i \in 0, \dots, N$, $V_i \in \kappa_i$ as :

$$\begin{aligned} V_N &= H, \\ V_i &= \mathbb{E}[H - \sum_{j=i}^{N-1} \nu_j \Delta S_j + \lambda \sum_{j=i}^{N-1} |\Delta \nu_{j-1}| S_{t_j} \mid \mathcal{F}_{t_i}], \forall i = 0, \dots, N-1. \end{aligned} \quad (3.2)$$

then

$$\begin{aligned} \mathbb{E}[(H - x - \hat{G}_T(\nu))^2] = & \mathbb{E}[(V_N - \nu_{N-1}\Delta S_{N-1} + \lambda|\Delta\nu_{N-2}|S_{t_{N-1}} - V_{N-1}) + \\ & \sum_{i=2}^{N-1} (V_i + \lambda|\Delta\nu_{i-2}|S_{t_{i-1}} - \nu_{i-1}\Delta S_{i-1} - V_{i-1}) + \\ & (V_1 + \lambda|\nu_0|S_{t_0} - \nu_0\Delta S_0 - x)^2] \end{aligned}$$

Due to the definition (3.2), we have that

$$E[V_i + \lambda|\Delta\nu_{i-2}|S_{t_{i-1}} - \nu_{i-1}\Delta S_{i-1} - V_{i-1}|\mathcal{F}_{t_i}] = 0, \forall i = 1, \dots, N,$$

so that

$$\begin{aligned} \mathbb{E}[(H - x - \hat{G}_T(\nu))^2] = & \mathbb{E}[\mathbb{E}[(V_N - \nu_{N-1}\Delta S_{N-1} + \lambda|\Delta\nu_{N-2}|S_{t_{N-1}} - V_{N-1})^2 | \mathcal{F}_{t_{N-1}}] + \\ & \mathbb{E}[\sum_{i=2}^{N-1} (V_i + \lambda|\Delta\nu_{i-2}|S_{t_{i-1}} - \nu_{i-1}\Delta S_{i-1} - V_{i-1})^2 + \\ & (V_1 + \lambda|\nu_0|S_{t_0} - \nu_0\Delta S_0 - x)^2] \end{aligned}$$

and iterating the process gives

$$\begin{aligned} \mathbb{E}[(H - x - \hat{G}_T(\nu))^2] = & \mathbb{E}[(V_N - \nu_{N-1}\Delta S_{N-1} + \lambda|\Delta\nu_{N-2}|S_{t_{N-1}} - V_{N-1})^2] + \\ & \sum_{i=2}^{N-1} \mathbb{E}[(V_i + \lambda|\Delta\nu_{i-2}|S_{t_{i-1}} - \nu_{i-1}\Delta S_{i-1} - V_{i-1})^2] + \\ & \mathbb{E}[(V_1 + \lambda|\nu_0|S_{t_0} - \nu_0\Delta S_0 - x)^2] \end{aligned}$$

Then we can write the problem (3.1) as:

$$\begin{aligned} \hat{\nu} = & \arg \min_{\nu \in \hat{\Theta}(\bar{m}, \bar{l})} \mathbb{E}[(V_N - \nu_{N-1}\Delta S_{N-1} + \lambda|\Delta\nu_{N-2}|S_{t_{N-1}} - V_{N-1})^2] + \\ & \sum_{i=2}^{N-1} \mathbb{E}[(V_i + \lambda|\Delta\nu_{i-2}|S_{t_{i-1}} - \nu_{i-1}\Delta S_{i-1} - V_{i-1})^2] + \\ & \mathbb{E}[(V_1 + \lambda|\nu_0|S_{t_0} - \nu_0\Delta S_0 - x)^2] \end{aligned} \quad (3.3)$$

The formulation (3.3) can be used to solve the problem by dynamic programming when the price process is Markov and the payoff a function of the asset value at maturity by introducing $\tilde{V}(t_i, S_{t_i}, \nu_{i-1})$ corresponding the optimal initial value of cash needed to minimize the future mean variance risk at date t_i with an asset value S_{t_i} for an investment ν_{i-1} chosen at date t_{i-1} .

The corresponding algorithm 1 is given introducing

$$\rho_i^{\bar{m}, \bar{l}}(\eta) = \{(V, \nu) / V, \nu \text{ are } \mathbb{R} \text{ valued } \mathcal{F}_{t_i}\text{-adapted with } -\bar{m} \leq \nu - \eta \leq \bar{l}\}.$$

Algorithm 1 Backward resolution for \mathcal{L}^2 minimization problem with iterated conditional expectation approximation.

- 1: $V(t_N, S_{t_N}(\omega), \nu_{N-1}) = H(\omega), \quad \forall \nu_{N-1}$
- 2: $R(t_N, S_{t_N}(\omega), \nu_{N-1}) = 0, \quad \forall \nu_{N-1}$
- 3: **for** $i = N, 2$ **do**
- 4:

$$(\tilde{V}(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}), \nu_{i-1}) = \arg \min_{(V, \nu) \in \rho_{i-1}^{\bar{m}, \bar{l}}(\nu_{i-2})} \mathbb{E}[(V(t_i, S_{t_i}, \nu) - \nu \Delta S_{i-1} + \lambda |\nu - \nu_{i-2}| S_{t_{i-1}} - V)^2 + R(t_i, S_{t_i}, \nu) | \mathcal{F}_{t_{i-1}}] \quad (3.4)$$

- 5: $R(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}) = \mathbb{E}[(V(t_i, S_{t_i}, \nu_{i-1}) - \nu_{i-1} \Delta S_{i-1} + \lambda |\Delta \nu_{i-2}| S_{t_{i-1}} - V(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}))^2 + R(t_i, S_{t_i}, \nu_{i-1}) | \mathcal{F}_{t_{i-1}}]$
 - 6: $\nu_0 = \arg \min_{\nu \in [-\bar{m}, \bar{l}]} \mathbb{E}[(V(t_1, S_{t_1}, \nu) + \lambda |\nu| S_{t_0} - \nu \Delta S_0 - x)^2 + R(t_1, S_{t_1}, \nu)]$
-

Remark 3.1. *The previous algorithm can be easily modified to solve local risk minimization problem of Schweitzer [29] where some liquidity constraints have been added. The equation (3.4) has to be modified by*

$$(\tilde{V}(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}), \nu_{i-1}) = \arg \min_{(V, \nu) \in \rho_{i-1}^{\bar{m}, \bar{l}}(\nu_{i-2})} \mathbb{E}[(V(t_i, S_{t_i}, \nu_{i-1}) - \nu \Delta S_{i-1} + \lambda |\nu - \nu_{i-2}| S_{t_{i-1}} - V)^2 | \mathcal{F}_{t_{i-1}}].$$

Due to approximation errors linked to the methodology used to estimate conditional expectation, the R estimation in algorithm 1 is prone to an error accumulation during the time iterations. Similarly to the scheme introduced in [3] to improve the methodology proposed in [17] to solve Backward Stochastic Differential Equations, we can propose a second version of the previous algorithm where the update for \bar{R} is taken ω by ω and stores the optimal trading gain function on each trajectory.

Algorithm 2 Backward resolution for \mathcal{L}^2 minimization problem avoiding conditional expectation iteration .

- 1: $\bar{R}(t_N, S_{t_{N-1}}(\omega), \nu_{N-1}) = H(\omega), \quad \forall \nu_{N-1}$
- 2: **for** $i = N, 2$ **do**
- 3:

$$(\tilde{V}(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}), \nu_{i-1}) = \arg \min_{(V, \nu) \in \rho_{i-1}^{\bar{m}, \bar{l}}(\nu_{i-2})} \mathbb{E}[\bar{R}(t_i, S_{t_i}, \nu) - \nu \Delta S_{i-1} + \lambda |\nu - \nu_{i-2}| S_{t_{i-1}} - V)^2 | \mathcal{F}_{t_{i-1}}] \quad (3.5)$$

- 4: $\bar{R}(t_{i-1}, S_{t_{i-1}}, \nu_{i-2}) = \bar{R}(t_i, S_{t_i}, \nu_{i-1}) - \nu_{i-1} \Delta S_{i-1} + \lambda |\Delta \nu_{i-2}| S_{t_{i-1}}$
 - 5: $\nu_0 = \arg \min_{\nu \in [-\bar{m}, \bar{l}]} \mathbb{E}[(\bar{R}(t_1, S_{t_1}, \nu) + \lambda |\nu| S_{t_0} - \nu \Delta S_0 - x)^2]$
-

Remark 3.1. *In order to treat the case of mean variance hedging that consists in finding the optimal strategy and the initial wealth to hedge the contingent claim the last line of algorithm 1 is replaced by*

$$(\tilde{V}, \nu_0) = \arg \min_{(V, \nu)} \mathbb{E}[(V(t_1, S_{t_1}, \nu) + \lambda|\nu|S_{t_0} - \nu\Delta S_0 - V)^2 + R(t_1, S_{t_1}, \nu)],$$

and last line of algorithm 2 by

$$(\tilde{V}, \nu_0) = \arg \min_{(V, \nu) \in \mathbb{R} \times [-\bar{m}, \bar{l}]} \mathbb{E}[(\bar{R}(t_1, S_{t_1}, \nu) + \lambda|\nu|S_{t_0} - \nu\Delta S_0 - V)^2].$$

3.2 Practical algorithm based on algorithm 2

Starting from the theoretical algorithm 2, we aim at getting an effective implementation based on a representation of the function \tilde{V} depending on time, S_t and the position ν_t in the hedging assets.

- In order to represent the dependency in the hedging position we introduce a time dependent grid

$$\mathcal{Q}_i := (\xi k)_{k=-(i+1)\lfloor \frac{\bar{m}}{\xi} \rfloor, \dots, (i+1)\lfloor \frac{\bar{l}}{\xi} \rfloor}$$

where ξ is the mesh size associated to the set of grids $(\mathcal{Q}_i)_{i=0, N}$ and, if possible, chosen such that $\frac{\bar{l}}{\xi} = \lfloor \frac{\bar{l}}{\xi} \rfloor$ and $\frac{\bar{m}}{\xi} = \lfloor \frac{\bar{m}}{\xi} \rfloor$.

- To represent the dependency in S_t we will use a Monte Carlo method using simulated path $\left((S_{t_i}^{(j)})_{i=0, \dots, N} \right)_{j=1, \dots, M}$ and calculate the arg min in equation (3.5) using a methodology close to the one described in [8]: suppose that we are given at each date t_i $(D_q^i)_{q=1, \dots, Q}$ a partition of $[\min_{j=1, M} S_{t_i}^{(j)}, \max_{j=1, M} S_{t_i}^{(j)}]$ such that each cell contains the same number of samples. We use the Q cells $(D_q^i)_{q=1, \dots, Q}$ to represent the dependency of \tilde{V} and ν in the S_{t_i} variable.

On each cell q we search for \hat{V}^q a linear approximation of the function \tilde{V} at a given date t_i and for a position $k\xi$ so that $\hat{V}^q(t_i, S, k) = a_i^q + b_i^q S$ is an approximation of $\tilde{V}(t_i, S, k\xi)$. On the cell q the optimal numerical hedging command $\hat{\nu}^q(k)$ for a position $k\xi$ can be seen as a sensibility so it is natural to search for a constant control per cell q when the value function is represented as a linear function.

Let us note $(l_i^q(j))_{j=1, \frac{M}{Q}}$ the set of all samples belonging to the cell q at date t_i . On each mesh the optimal control $\hat{\nu}^q$ is obtained by discretizing the command ν on a grid $\eta = ((k+r)\xi)_{r=-\lfloor \frac{\bar{m}}{\xi} \rfloor, \dots, \lfloor \frac{\bar{l}}{\xi} \rfloor}$, and by testing the one giving a \hat{V}^q value minimizing the \mathcal{L}^2 risk so solving equation (3.5).

The algorithm 3 permits to find the optimal $\nu_i^{(j)}(k)$ command using algorithm 2 at date t_i , for a hedging position $k\xi$ and for all the Monte Carlo simulations j . For each command tested on the cell q the corresponding \hat{V}^q function is calculated by regression.

Algorithm 3 Optimize minimal hedging position $(\hat{\nu}_{t_i}^{(l)}(k))_{l=1,\dots,M}$ at date t_{i-1}

```

1: procedure OPTIMALCONTROL(  $\bar{R}(t_{i+1}, \cdot, \cdot), k, S_{t_i}, S_{t_{i+1}}$  )
2:   for  $q = 1, Q$  do
3:      $P = \infty,$ 
4:     for  $k = -\lfloor \frac{\bar{m}}{\xi} \rfloor, \dots, \lfloor \frac{\bar{l}}{\xi} \rfloor$  do
5:        $(a_i^q, b_i^q) = \arg \min_{(a,b) \in \mathbb{R}^2} \sum_{j=1}^{\frac{M}{Q}} (\bar{R}(t_{i+1}, S_{t_{i+1}}^{l_i^q(j)}, (k+l)\xi) -$ 
         $(k+l)\xi \Delta S_{t_i}^{l_i^q(j)} +$ 
         $\lambda |l\xi| S_{t_i}^{l_i^q(j)} - (a + b S_{t_i}^{l_i^q(j)}))^2$ 
6:        $\tilde{P} = \sum_{j=1}^{\frac{M}{Q}} (\bar{R}(t_{i+1}, S_{t_{i+1}}^{l_i^q(j)}, (k+l)\xi) - (k+l)\xi \Delta S_{t_i}^{l_i^q(j)} +$ 
         $\lambda |l\xi| S_{t_i}^{l_i^q(j)} - (a_i^q + b_i^q S_{t_i}^{l_i^q(j)}))^2$ 
7:       if  $\tilde{P} < P$  then
8:          $\nu^q = k\xi, P = \tilde{P}$ 
9:       for  $j = 1, \frac{M}{Q}$  do
10:         $\hat{\nu}_i^{(l_i^q(j))}(k) = \nu^q$ 
return  $(\hat{\nu}_{t_i}^{(j)}(k))_{j=1,\dots,M}$ 

```

Remark 3.2. *It is possible to use different discretization ξ to define the set η and the set \mathcal{Q}_i . Then an interpolation is needed to get the \bar{R} values at a position not belonging to the grid. An example of the use of such an interpolation for gas storage problem tracking the optimal cash flow generated along the Monte Carlo strategies can be found in [32].*

Remark 3.3. *This algorithm permits to add some global constraint on the global liquidity of the hedging asset. This is achieved by restricting the possible hedging positions to a subset of \mathcal{Q}_i at each date t_i .*

Then the global discretized version of algorithm 2 is given on algorithm 4 where $H^{(j)}$ correspond to the j th Monte Carlo realization of the payoff.

Algorithm 4 Global backward resolution algorithm, optimal control and optimal variance calculation

```

1: for  $\nu \in \mathcal{Q}_{N-1}$  do
2:   for  $j \in [1, M]$  do
3:      $\bar{R}(t_N, S_{t_N}^{(j)}, \nu) = H^{(j)}$ 
4:   for  $i = N, 2$  do
5:     for  $k\xi \in \mathcal{Q}_{i-2}$  do
6:        $(\nu_{i-1}^{(j)}(k))_{j=1, M} = \text{OptimalControl}(\bar{R}(t_i, \cdot, \cdot), k, S_{t_{i-1}}, S_{t_i}),$ 
7:       for  $j \in [1, M]$  do
8:          $\bar{R}(t_{i-1}, S_{t_{i-1}}^{(j)}, k\xi) = \bar{R}(t_i, S_{t_i}^{(j)}, \nu_{i-1}^{(j)}(k)) -$ 
            $\nu_{i-1}^{(j)}(k)\Delta S_{i-1}^{(j)} + \lambda|\nu_{i-1}^{(j)}(k) - k\xi|S_{t_{i-1}}^{(j)}$ 
9:    $P = \infty,$ 
10:  for  $k = -\lfloor \frac{\bar{m}}{\bar{\xi}} \rfloor, \dots, \lfloor \frac{\bar{l}}{\bar{\xi}} \rfloor$  do
11:     $\tilde{P} = \sum_{j=1}^M (\bar{R}(t_1, S_{t_1}^{(j)}, k\xi) - k\xi\Delta S_0^{(j)} + \lambda|k|\xi S_0 - x)^2$ 
12:    if  $\tilde{P} < P$  then
13:       $\nu_0 = k\xi, P = \tilde{P}$ 
14:   $Var = \frac{1}{M} \sum_{j=1}^M (\bar{R}(t_1, S_{t_1}^{(j)}, \nu_0) - \nu_0\Delta S_0^{(j)} + \lambda|\nu_0|S_0 - x)^2$ 

```

4 Numerical results

4.1 The uncertainty models and the problem to solve

We suppose that an electricity retailer has to face uncertainty on the load he has to provide for his customers. We suppose that this load given at $O(t)$ is stochastic and follows the dynamic:

$$O(t) = \hat{O}(t) + (O(u) - \hat{O}(u))e^{-a_O(t-u)} + \int_u^t \sigma_O e^{-a_O(t-s)} dW_s^O, \quad (4.1)$$

where a_O is a mean-reverting coefficient, σ_O the volatility of the process, $(W_t^O)_{t \leq T}$ is a Brownian on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{O}(u)$ is the average load seen on the previous years at the given date u . The equation (4.1) only states that the load curve $O(t)$ oscillates around an average value $\hat{O}(t)$ due to economic activity and to thermosensibility.

We suppose that the retailer wants to hedge his position for a given date T and that the future model is given by a one factor HJM model under the real world probability

$$\begin{aligned}
F(t, T) &= F(0, T) e^{-\frac{V(t, T)}{2} + e^{-a_E(T-t)} \hat{W}_t^E}, \\
V(t, T) &= \sigma_E^2 \frac{e^{-2a_E(T-t)} - e^{-2a_E T}}{2a_E}, \\
\hat{W}_t^E &= \sigma_E \int_0^t e^{-a_E(t-s)} dW_s^E,
\end{aligned} \quad (4.2)$$

where $F(t, T)$ is the forward curve seen at date t for delivery at date T , a_E the mean reverting parameter for electricity σ_E the volatility of the model and $(W_t^E)_{t \leq T}$ a Brownian on $(\Omega, \mathcal{F}, \mathbb{P})$ correlated to W^O with a correlation ρ . The correlation is a priori negative, indicating that a high open position which is a signal of high available production or low consumption drives the prices down.

Remark 4.1. *The fact that the future price is modeled as a martingale is linked to the fact the risk premium is difficult to estimate and of second order compared to the volatility of the asset.*

The SDE associated to the model (4.2) is

$$dF(t, T) = \sigma_E e^{-a_E(T-t)} F(t, T) dW_t^E.$$

Using equation 4.2, we get for

$$\mathbb{E}\left[\frac{F(t_i, T)^2}{F(t_{i-1}, T)^2} \middle| \mathcal{F}_{t_{i-1}}\right] = e^{\sigma_E^2 \frac{e^{-2a_E(T-t_i)} - e^{-2a_E(T-t_{i-1})}}{2a_E}}$$

so that assumption 2.4 is satisfied.

Similarly

$$\mathbb{E}\left[\frac{F(t_{i-1}, T)^2}{F(t_i, T)^2} \middle| \mathcal{F}_{t_{i-1}}\right] = e^{3\sigma_E^2 \frac{e^{-2a_E(T-t_i)} - e^{-2a_E(T-t_{i-1})}}{2a_E}}$$

so that assumption 2.5 is satisfied. At last assumption 2.6 is satisfied taking

$$\delta = \max_{i=1}^N e^{-\sigma_E^2 \frac{e^{-2a_E(T-t_i)} - e^{-2a_E(T-t_{i-1})}}{2a_E}} < 1$$

Remark 4.2. *On a more realistic test case, the load curve is given for daily delivery and the future products corresponds to monthly product. The retailer wants to hedge until the beginning of the delivery month with monthly products and then has to face daily netting. Supposing that the daily dynamic for daily delivery is given by a one factor HJM model, the dynamic of the monthly curve can be reconstructed numerically and this product will be used to define the optimal hedging policy. Examples of such aggregation are given in [32].*

The payoff of such a contract is then $H := O(T)F(T, T)$. When there is no liquidity constraints so supposing that the portfolio re-balancing is continuous, with no fees and constraints on volume and using the mean variance criterion (see remark 3.1) the value of the contract $V = (V(t, O(t), F(t, T)))_{t \leq T}$ and the hedging policy $\nu = (\nu(t, O(t), F(t, T)))_{t \leq T}$ solve :

$$(V, \nu) = \arg \min_{(\tilde{V}, \tilde{\nu}) \in \mathbb{R} \times \mathcal{L}^2(F)} \mathbb{E}[(O(T)F(T, T) - \int_t^T \tilde{\nu}_s dF(s, T) - \tilde{V})^2 | \mathcal{F}_t] \quad (4.3)$$

where $\mathcal{L}^2(F)$ is the set of predictable process v satisfying

$$\mathbb{E}\left[\int_0^T v_t^2 \sigma_E^2 e^{-2a_E(T-t)} F(t, T)^2 dt\right] < \infty$$

The solution of the previous problem is known to be given in the case of martingale assets by the Galtchouk-Kunita-Wananabe decomposition and

$$V(t, O(t), F(t, T)) = \mathbb{E}[O(T)F(T, T)|\mathcal{F}_t],$$

which can be calculated as follows:

$$V(t, O(t), F(t, T)) = F(t, T)[\hat{O}(t) + (O(t) - \hat{O}(t))e^{-a_O(T-t)} + \rho\sigma_E\sigma_O \frac{1 - e^{-(a_E+a_O)(T-t)}}{a_E + a_O}].$$

The value function V being a martingale, using Ito lemma we get:

$$O(T)F(T, T) = V(0, O(0), F(0, T)) + \int_0^T \frac{\partial V}{\partial O}(s, O(s), F(s, T))\sigma_D dW_s^O + \int_0^T \frac{\partial V}{\partial F}(s, O(s), F(s, T))\sigma_E F(s, T)e^{-a_E(T-s)} dW_s^E,$$

so using $W_t^O = \rho W_t^E + \sqrt{1 - \rho^2}\hat{W}_t^O$ with \hat{W}_t^O orthogonal to W_t^E in \mathcal{L}^2 :

$$O(T)F(t, T) = V(0, O(0), F(0, T)) + \int_0^T \left(\frac{\partial V}{\partial F}(s, O(s), F(s, T)) + \rho \frac{\sigma_D e^{a_E(T-s)}}{\sigma_E F(s, T)} \frac{\partial V}{\partial O}(s, O(s), F(s, T)) \right) dF(s, T) + \sqrt{1 - \rho^2} \frac{\partial V}{\partial O}(s, O(s), F(s, T))\sigma_D d\hat{W}_s^O. \quad (4.4)$$

The second part in the previous integral represents the non hedgable part of the asset and the optimal hedge is given by:

$$\nu(t, O(t), F(t, T)) = \frac{\partial V}{\partial F}(t, O(t), F(t, T)) + \rho \frac{\sigma_D e^{a_E(T-t)}}{\sigma_E F(t, T)} \frac{\partial V}{\partial O}(t, O(t), F(t, T)).$$

Introducing the forward tangent process

$$Y_t^T = e^{-\frac{V(t, T)}{2} + e^{-a_E(T-t)}\hat{W}_t^E},$$

we get

$$\begin{aligned} \frac{\partial V}{\partial F}(t, O(t), F(t, T)) &= \mathbb{E}(O(T)Y_T^T | \mathcal{F}_t) / Y_t^T, \\ \frac{\partial V}{\partial O}(t, O(t), F(t, T)) &= e^{-a_O(T-t)} F(t, T), \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial V}{\partial F}(t, O(t), F(t, T)) &= \hat{O}(t) + (O(t) - \hat{O}(t))e^{-a_O(T-t)} + \\ &\rho\sigma_E\sigma_O \frac{1 - e^{-(a_E+a_O)(T-t)}}{a_E + a_O}. \end{aligned}$$

The optimal hedging policy can then be rewritten as

$$\begin{aligned} \nu(t, O(t), F(t, T)) = & \hat{O}(t) + (O(t) - \hat{O}(t))e^{-a_O(T-t)} \\ & + \rho \left[e^{(a_E - a_O)(T-t)} \frac{\sigma_O}{\sigma_E} + \sigma_E \sigma_O \frac{1 - e^{-(a_E + a_O)(T-t)}}{a_E + a_O} \right]. \end{aligned} \quad (4.5)$$

Traders often hedge their risks only using the sensibility of the option with respect to the underlying getting a non optimal hedging policy $\tilde{\nu}$ for the mean variance criterion:

$$\begin{aligned} \tilde{\nu}(t, O(t), F(t, T)) = & \frac{\partial V}{\partial F}(t, O(t), F(t, T)) \\ = & \hat{O}(t) + (O(t) - \hat{O}(t))e^{-a_O(T-t)} + \rho \sigma_E \sigma_O \frac{1 - e^{-(a_E + a_O)(T-t)}}{a_E + a_O}. \end{aligned} \quad (4.6)$$

4.2 The test case

We suppose that a big producer wants to hedge from January 2016 an average open position position of $8.5Gw$ for January 2017 using January monthly products or OTC January products when market products are not available. This open position evolves according (4.1) and we take two set of parameters:

- Set A of parameters given annually: $a_O = 0.1$, $\sigma_O = 1300$ so that we get a high mean reversion characteristic. These parameters are typical parameters associated to thermosensibility.
- Set B of parameters given annually: $a_0 = 0.024$, $\sigma_O = 640$ chosen such that the equivalent volatility at the beginning of the study of the curve at delivery date January 2017 is the same as for the set A. These parameters correspond to slower movement of the curve.

The different scenarios of the load curve are given in figure 2.

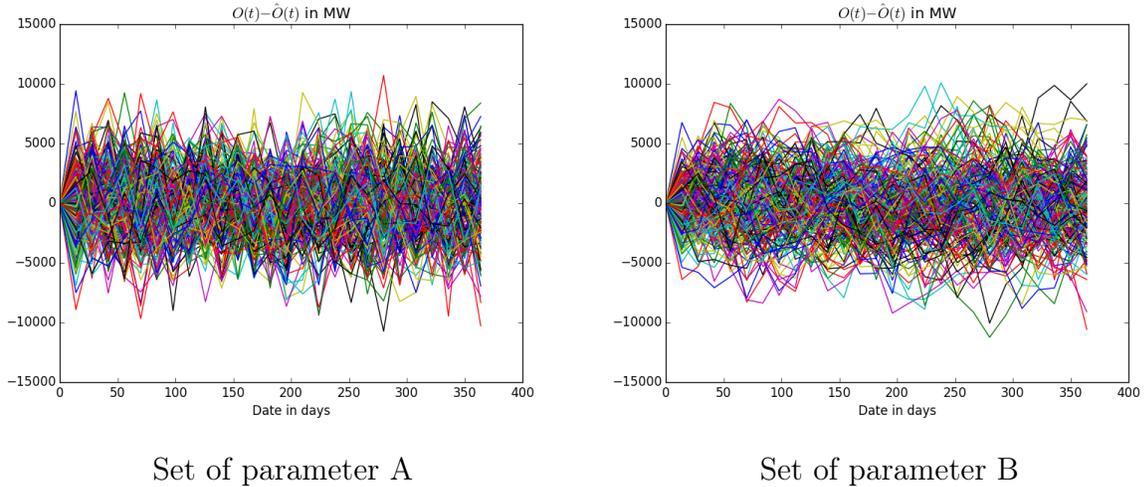
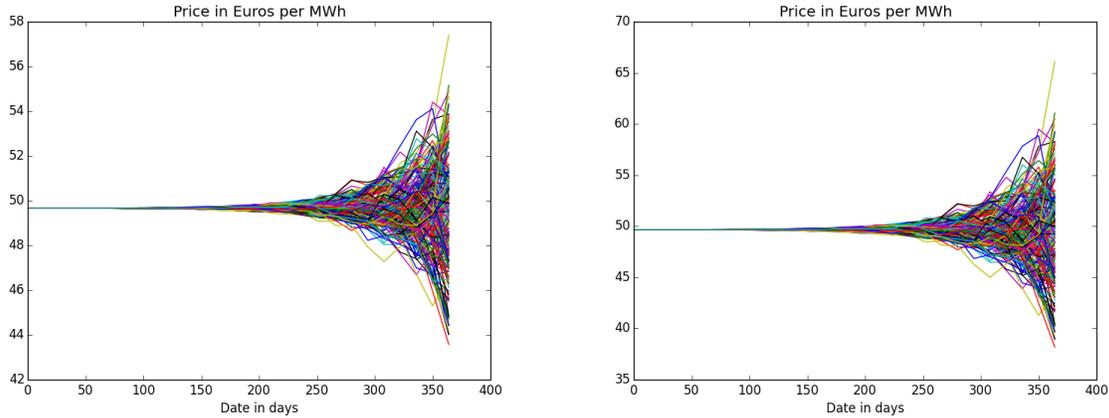


Figure 1: $O(t) - \hat{O}(t)$ scenarios during 2016.

We take two sets of parameters for the price dynamic (4.2).

- Set C of parameters with $a_E = 5$, $\sigma_E = 20\%$ annually obtained using the month dynamic in the long term,
- Set D of parameters multiplying the volatility by 2 and keeping the same mean reverting coefficient.

The resulting scenarios obtained are given on figure 2.



Future price scenarios January 2017 with low volatility Future price scenarios for January 2017 with high volatility

Figure 2: Price scenarios of January 2017 monthly product from January 2016 for parameters set C and D.

We suppose that the transition cost are zero so we are only interested in the effect of the frequency of the hedge and the depth of the future market. We will test the hedging strategies in term of variance with 160000 simulations corresponding to:

- the optimal analytic given by equation (4.5),
- the classical tangent delta given by equation (4.6),
- the numerical optimal solution obtained by algorithm 4 implemented with the StOpt library [16] where the optimal strategy has been calculated using 160000 trajectories and 12×6 meshes following [8], where 12 is the number of meshes for the price representation and 6 is the number of meshes for the load representation.

Remark 4.3. Here the value function is a function of F and O so that two dimensional regressions have to be used in the algorithm 4 leading to the need of the specification of the number of meshes used for F and O representation.

In all the tests, we suppose that hedging is achieved twice a month.

4.2.1 Market with infinite depth

Supposing infinite market depth for the monthly future product, we will look at the effect of the hedge frequency and the correlation ρ .

We only test the set of parameters C for the prices combining the set of parameters A and B for the load curve dynamic in table 1. When there is no correlation, all the three strategies give the same results for the set of parameters A and B.

On set of parameters A and B, the use of the optimal analytic hedge permits to highly reduce the standard deviation obtained by the tangent delta.

On the set of parameter B, no further reduction is obtained using the numerical method indicating that with this kind of slow dynamic of the load curve a hedge achieved once every two weeks is sufficient.

On the set of parameter A, as the correlation increases, reduction in term of standard deviation is obtained using the numerical control instead of the optimal analytic indicating that on this rapid varying load curve we have to take into account the hedge frequency.

Parameter	A	A	A	A	B	B	B	B
ρ	0.	-0.2	-0.4	-0.6	0.	-0.2	-0.4	-0.6
Tangent delta	7.31	7.28	7.25	7.22	9.58	9.55	9.52	9.5
Optimal analytic	7.31	7.18	6.88	6.4	9.58	9.37	8.8	8.05
Numeric optimal	7.32	7.16	6.76	6.24	9.59	9.39	8.82	8.05

Table 1: Standard deviation obtained for infinite market depth, set of parameter C for the prices.

4.3 Market with a finite depth for the future available

We suppose here a realistic depth of 500 MW available of January 2017 product at each hedging date such that at each hedging date we can buy or sell up to 500 MW. On table 2 and 3, we give the standard deviation observed with depth constraint for the two set of price parameters C and D. Once again for the set of parameters B, the numerical method and the optimal analytic one give the same results improving a lot the results obtained by the tangent delta hedging as soon as the correlation is non zero. On the case A, once again taking into account the finite depth with the numerical highly improves the results when correlation is not zero.

Parameter	A	A	A	A	B	B	B	B
ρ	0.	-0.2	-0.4	-0.6	0.	-0.2	-0.4	-0.6
Tangent delta	7.31	7.28	7.25	7.22	9.58	9.55	9.52	9.49
Optimal analytic	7.31	7.24	7.16	7.07	9.58	9.37	8.81	8.05
Numeric optimal	7.31	7.18	6.89	6.4	9.58	9.37	8.81	8.05

Table 2: Standard deviation obtained for low market depth, set of parameter C for the prices.

Parameter	A	A	A	A	B	B	B	B
ρ	0.	-0.2	-0.4	-0.6	0.	-0.2	-0.4	-0.6
Tangent delta	7.38	7.32	7.27	7.21	9.63	9.58	9.53	9.49
Optimal analytic	7.38	7.27	7.12	6.96	9.63	9.4	8.8	7.74
Numeric optimal	7.38	7.21	6.87	6.32	9.64	9.4	8.8	7.74

Table 3: Standard deviation obtained for low market depth, set of parameter D for the prices.

On figure 3 and 4, we give the strategies obtained with the set of parameter D for the price without and with a correlation -0.2. The strategy obtained by the numerical method are more volatile and more cautious than with analytic methods: the optimal numerical strategy tends to sell less in order to avoid to buy again energy near the delivery date. The optimal analytic solution often try to buy again energy near maturity but cannot match the desired energy analytic target because of the depth constraint. As seen on figure 4, the two analytic solutions are not very sensitive to the correlation but the numerical one tends to sell far less in presence of correlation.

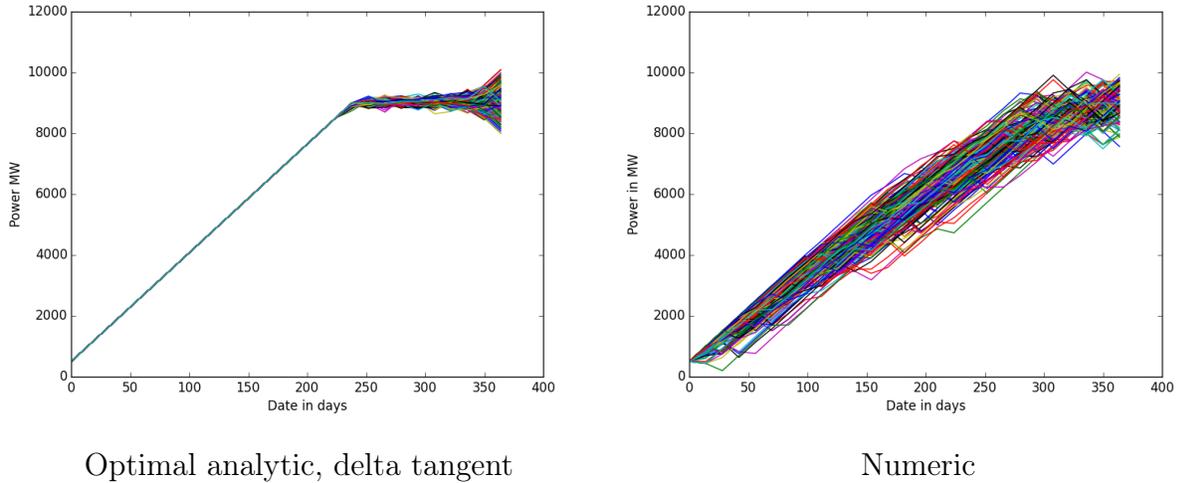
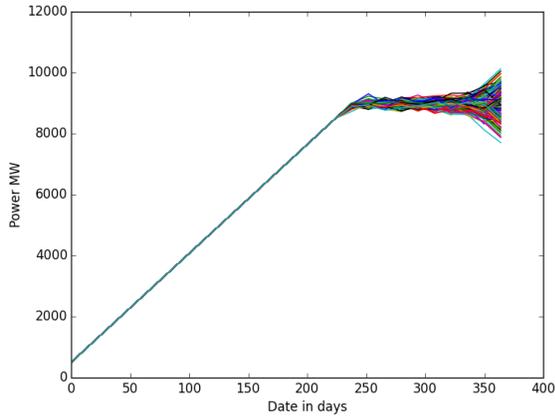
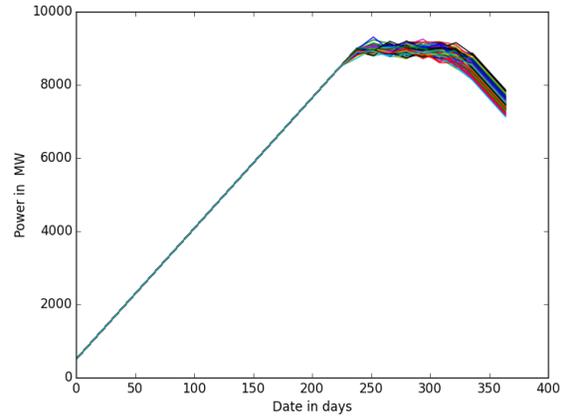


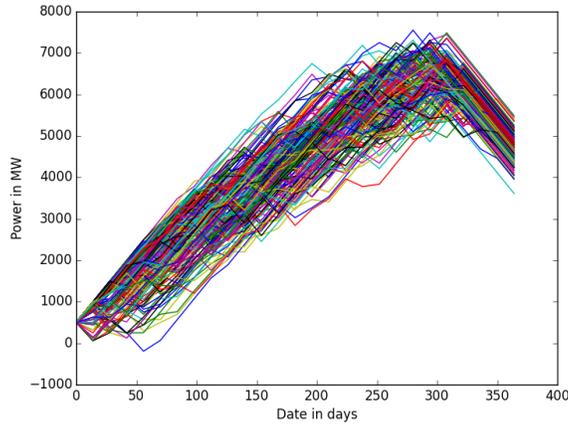
Figure 3: Hedging position simulations in MW set of parameter A for load, D for electricity, low market depth, no correlation.



delta tangent



analytic optimal



Numeric

Figure 4: Hedging simulations in MW with set of parameter A for load, D for electricity, low market depth and correlation -0.2.

5 Conclusion

In the case of mean variance hedging of the wealth portfolio an effective algorithm has been developed to find the optimal strategy taking into account the transition cost, the limited availability of the hedging product, the fact that hedging is only achieved at discrete dates. We have shown on a realistic case in energy market that taking in account the reality of the market depth has an important impact on the efficiency of the hedging strategy.

This algorithm could be extended to non symmetric risk measure to take into account the fact that managers wants to favor gains.

References

- [1] René Aid, Luciano Campi, and Nicolas Langrené. A structural risk-neutral model for pricing and hedging power derivatives. *Mathematical Finance*, 23(3):387–438, 2013.
- [2] Martin T Barlow. A diffusion model for electricity prices. *Mathematical finance*, 12(4):287–298, 2002.
- [3] Christian Bender and Robert Denk. A forward scheme for backward sdes. *Stochastic processes and their applications*, 117(12):1793–1812, 2007.
- [4] Fred Espen Benth, Lars Ekeland, Ragnar Hauge, and Bjørn Fredrik Nielsen. A note on arbitrage-free pricing of forward contracts in energy markets. *Applied Mathematical Finance*, 10(4):325–336, 2003.
- [5] Fred Espen Benth and Jurate Saltyte-Benth. The normal inverse gaussian distribution and spot price modelling in energy markets. *International journal of theoretical and applied finance*, 7(02):177–192, 2004.
- [6] Dimitris Bertsimas, Leonid Kogan, and Andrew W Lo. When is time continuous? *Journal of Financial Economics*, 55(2):173–204, 2000.
- [7] Eric Beutner. Mean–variance hedging under transaction costs. *Mathematical Methods of Operations Research*, 65(3):539–557, 2007.
- [8] Bruno Bouchard and Xavier Warin. Monte-carlo valuation of american options: facts and new algorithms to improve existing methods. In *Numerical methods in finance*, pages 215–255. Springer, 2012.
- [9] René Carmona and Michael Coulon. A survey of commodity markets and structural models for electricity prices. In *Quantitative Energy Finance*, pages 41–83. Springer, 2014.
- [10] Aleš Černý. Dynamic programming and mean-variance hedging in discrete time. *Applied Mathematical Finance*, 11(1):1–25, 2004.
- [11] Aleš Černý and Jan Kallsen. Hedging by sequential regressions revisited. *Mathematical Finance*, 19(4):591–617, 2009.
- [12] Emmanuel Denis and Yuri Kabanov. Mean square error for the leland–lott hedging strategy: convex pay-offs. *Finance and Stochastics*, 14(4):625–667, 2010.
- [13] H Föllmer and Martin Schweizer. Hedging of contingent claims. *Applied stochastic analysis*, 5:389, 1991.
- [14] Masaaki Fukasawa et al. Discretization error of stochastic integrals. *The Annals of Applied Probability*, 21(4):1436–1465, 2011.

- [15] Jim Gatheral. No-dynamic-arbitrage and market impact. *Quantitative finance*, 10(7):749–759, 2010.
- [16] Hugo Gevret, Jerome Lelong, and Xavier Warin. *STochastic OPTimization library in C++*. PhD thesis, EDF Lab, 2016.
- [17] Emmanuel Gobet, Jean-Philippe Lemor, Xavier Warin, et al. A regression-based monte carlo method to solve backward stochastic differential equations. *The Annals of Applied Probability*, 15(3):2172–2202, 2005.
- [18] Takaki Hayashi and Per A Mykland. Evaluating hedging errors: an asymptotic approach. *Mathematical finance*, 15(2):309–343, 2005.
- [19] Farshid Jamshidian. Commodity option evaluation in the gaussian futures term structure model. *Review of Futures Markets*, 10:324–346, 1990.
- [20] Yuri M Kabanov and Mher M Safarian. On leland’s strategy of option pricing with transactions costs. *Finance and Stochastics*, 1(3):239–250, 1997.
- [21] Damien Lamberton, Huy en Pham, and Martin Schweizer. Local risk-minimization under transaction costs. *Mathematics of Operations Research*, 23(3):585–612, 1998.
- [22] Nicolas Landon. *Almost sure optimal stopping times: Theory and applications*. PhD thesis, Palaiseau, Ecole polytechnique, 2013.
- [23] Hayne E Leland. Option pricing and replication with transactions costs. *The journal of finance*, 40(5):1283–1301, 1985.
- [24] Francis A Longstaff and Eduardo S Schwartz. Valuing american options by simulation: a simple least-squares approach. *The review of financial studies*, 14(1):113–147, 2001.
- [25] M Motoczyński. Multidimensional variance-optimal hedging in discrete-time model – a general approach. *Mathematical Finance*, 10(2):243–257, 2000.
- [26] S Pergamenschikov et al. Limit theorem for leland’s strategy. *The Annals of Applied Probability*, 13(3):1099–1118, 2003.
- [27] Mathieu Rosenbaum, Peter Tankov, et al. Asymptotically optimal discretization of hedging strategies with jumps. *The Annals of Applied Probability*, 24(3):1002–1048, 2014.
- [28] Martin Schweizer. Variance-optimal hedging in discrete time. *Mathematics of Operations Research*, 20(1):1–32, 1995.
- [29] Martin Schweizer. A guided tour through quadratic hedging approaches. Technical report, Discussion Papers, Interdisciplinary Research Project 373: Quantification and Simulation of Economic Processes, 1999.

- [30] Halil M Soner, Steven E Shreve, and Jakša Cvitanić. There is no nontrivial hedging portfolio for option pricing with transaction costs. *The Annals of Applied Probability*, pages 327–355, 1995.
- [31] Peter Tankov and Ekaterina Voltchkova. Asymptotic analysis of hedging errors in models with jumps. *Stochastic processes and their applications*, 119(6):2004–2027, 2009.
- [32] Xavier Warin. Gas storage hedging. In *Numerical Methods in Finance*, pages 421–445. Springer, 2012.
- [33] Ruotao Zhang. *Couverture approchée des options Européennes*. PhD thesis, Ecole des Ponts ParisTech, 1999.