Cours Bachelier, Paris, France

# Game Theory: Fundamentals and Application to Wireless and Electricity Networks (Cours 1 à 4)

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# Point of view

- Start from scratch
- Broad audience
- Overview
- Methodologies
- Application-oriented
- Channels: 1. Slides; 2. Speech; 3. Board

# Outline

## Class $\#1$:

**0. Introduction**

**1. Strategic form games**

## Classes $\#2$ and $\#3$:

**2. Dynamic games**
2.1. Repeated games
2.2. Case study 1: Wireless power control
2.3. Feasible utility region and Shannon theory
2.4. Stochastic games, mean-field games, differential games
2.5. Case study 2: Consumption power scheduling

# Outline

## Class $\#4$:

**3. More**

3.X. Coalition form games

3.X. Learning in games

3.X. More examples: Game theory and finance?

3.X. Case study, viral marketing strategies

3.X. Bayesian games, signalling games

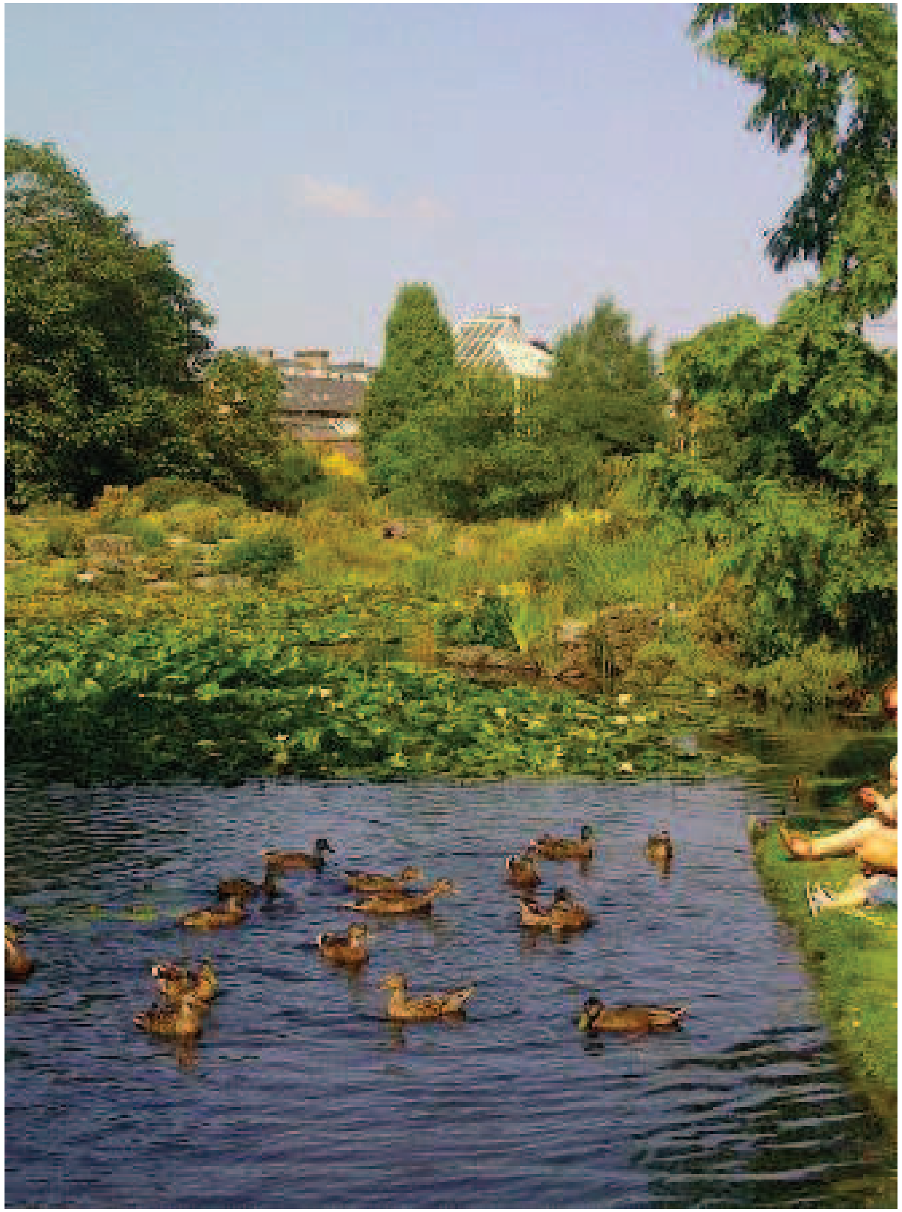
3.X. Proofs?

3.4. Conclusion

# 0. Introduction

Game Theory and Optimization...

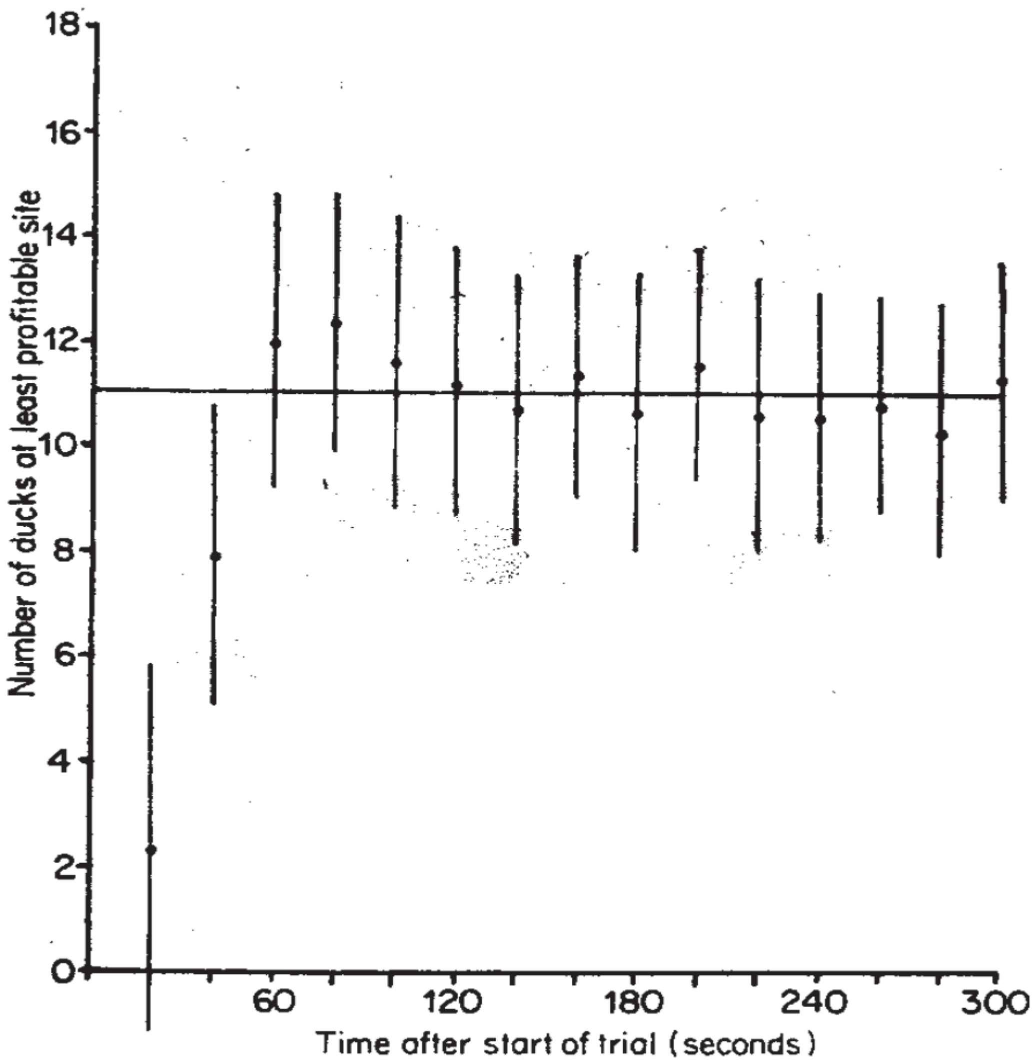
# Cambridge University, UK, winter 1979



## A couple of details about one of the experiments

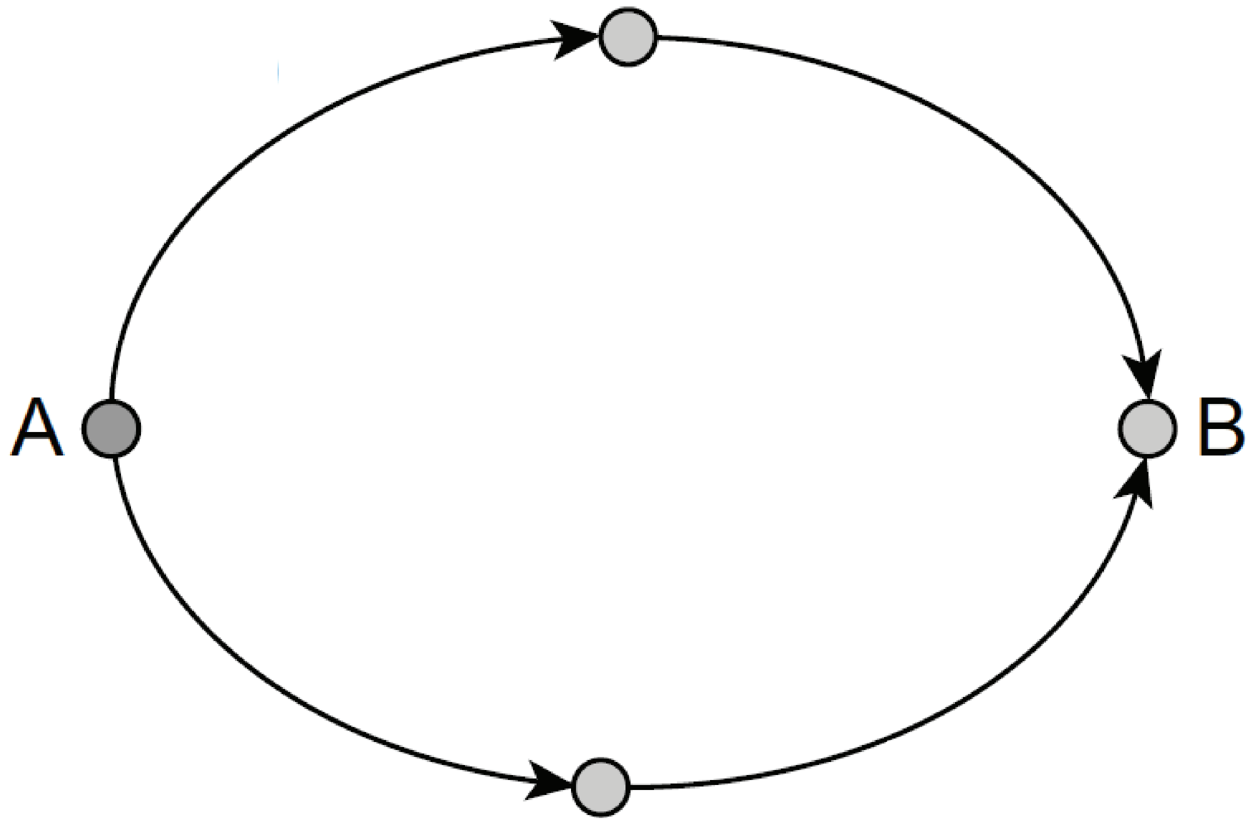
- 33 ducks.
- Two observers/sites 20 m apart.
- Site 1: 12 items/min.
- Site 2: 24 items/min.

# Observations

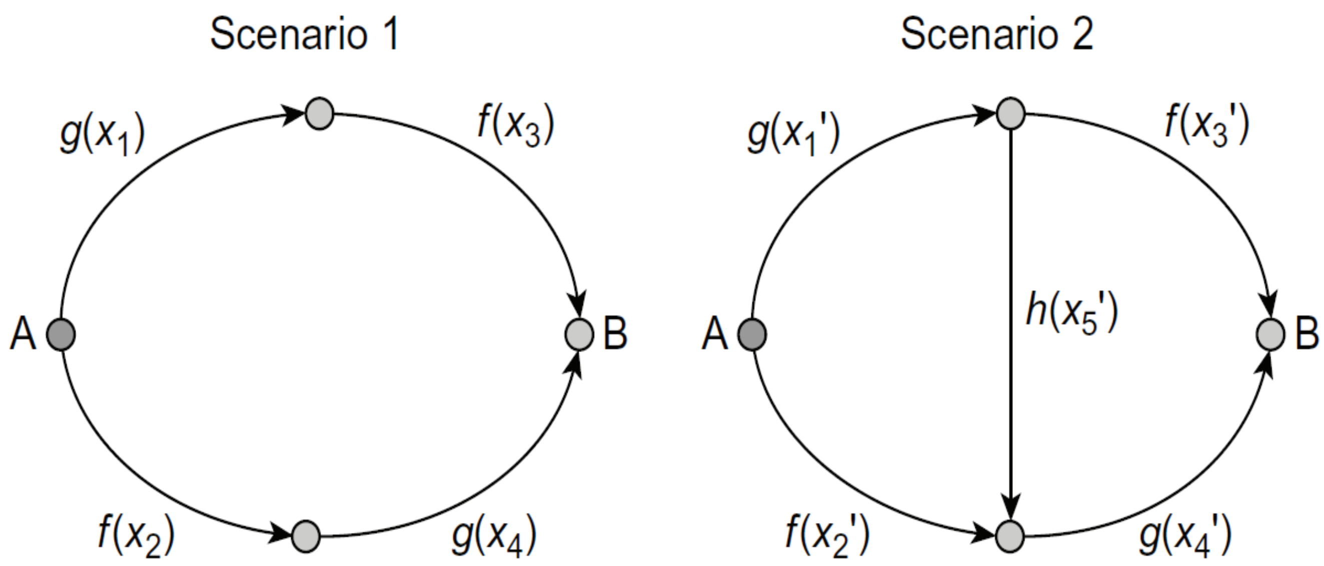



D. G. Harper, "Competitive foraging in mallards: Ideal free ducks", *Anim. Behav.*, 1982, 30, 575-585.

# Ducks become drivers



# Stuttgart, Germany, 1969 [Braess 1969]




| Input flow = 6 | |
|---|---|
| $f(x) = x + 50$, $g(x) = 10x$ | |
| $h(x) = +\infty$ | $h(x) = x + 10$ |
| 83 min | 92 min |
| $(x_1, x_2) = (3, 3)$ | $(x'_1, x'_2, x'_3) = (4, 2, 2)$ |

# Observing Braess-type instances

## In the real life

- Stuttgart 1969: investments into the road network $\Rightarrow$ traffic $\searrow$. Section of newly-built road closed $\Rightarrow$ traffic $\nearrow$ [Knödel 1969].
- NYC 1990: closing of 42nd street in New York City $\Rightarrow$ amount of congestion in the area $\nearrow$ [New York Times 1990].
- Seoul 2003: one of the three tunnels shut down to restore a river and a park $\Rightarrow$ traffic flow improved.

**In many other situations:** Wireless networks [Cohen and Kelly 1990][Perlaza et al 2009], energy networks [Baillieul et al 2015], ...

# About the paradox

## Trivial inequality in standard optimization

$$\max_{x \in \mathcal{A}} f(x) \leq \max_{x \in \mathcal{B}} f(x)$$

when $\mathcal{A} \subseteq \mathcal{B}$.

**This inequality does not hold anymore $\Leftarrow$ (partial control + multiple utility functions)**

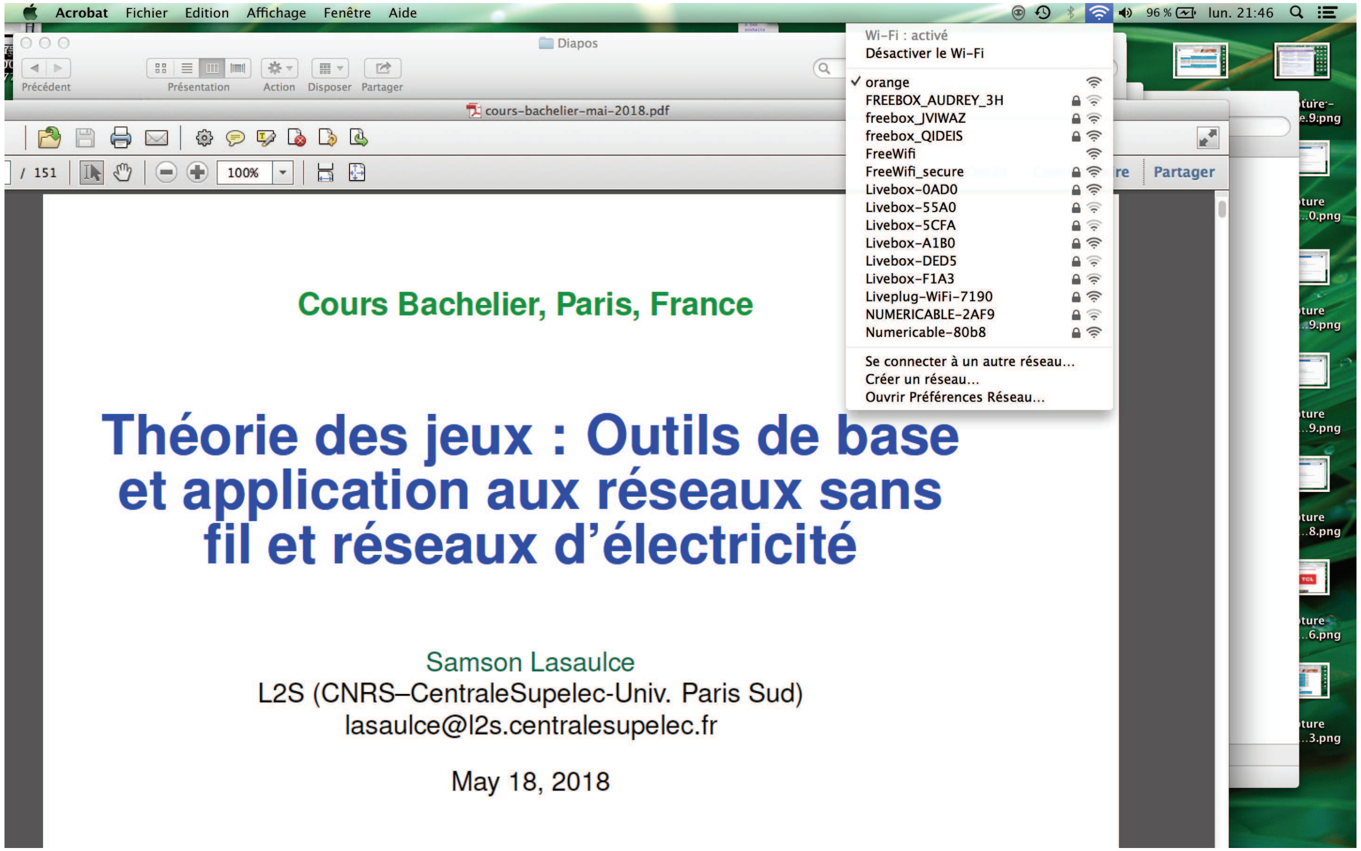
$$x = (x_1, ..., x_K).$$

# Why only partial control?

**Complexity** issues. ① Smart grid example: charging instant selection $48$ time-slots and $16$ vehicles $48^{16} > 32^{16} > 10^{21}$.

# Why only partial control?

**Complexity** issues. ② Wireless example: channel selection with $16$ channels and $16$ users $16^{16} = 2^{64} > 10^{18}$.



## Why multiple utility functions?

- Main function decomposition,
- several performance criteria,
- several decision-makers, ...

# Let's recap. DO-GT-MOO

► Distributed optimization (DO): typically about partial control with one DM.

► Multi-objective optimization (MOO): typically about one DM with full control + several objectives [Björnson et al 2015].

► ”Non-cooperative” game theory (GT): typically about several (virtual/real) DMs with partial control + several objectives.
What is the meaning of optimality then?

# Typical issues in scenarios with partial control and multiple objectives

► Which solution concept to consider as a possible game outcome?

► Does it exist for the game of interest? Is it unique?

► Is it efficient? How do we measure efficiency? How do we improve it?

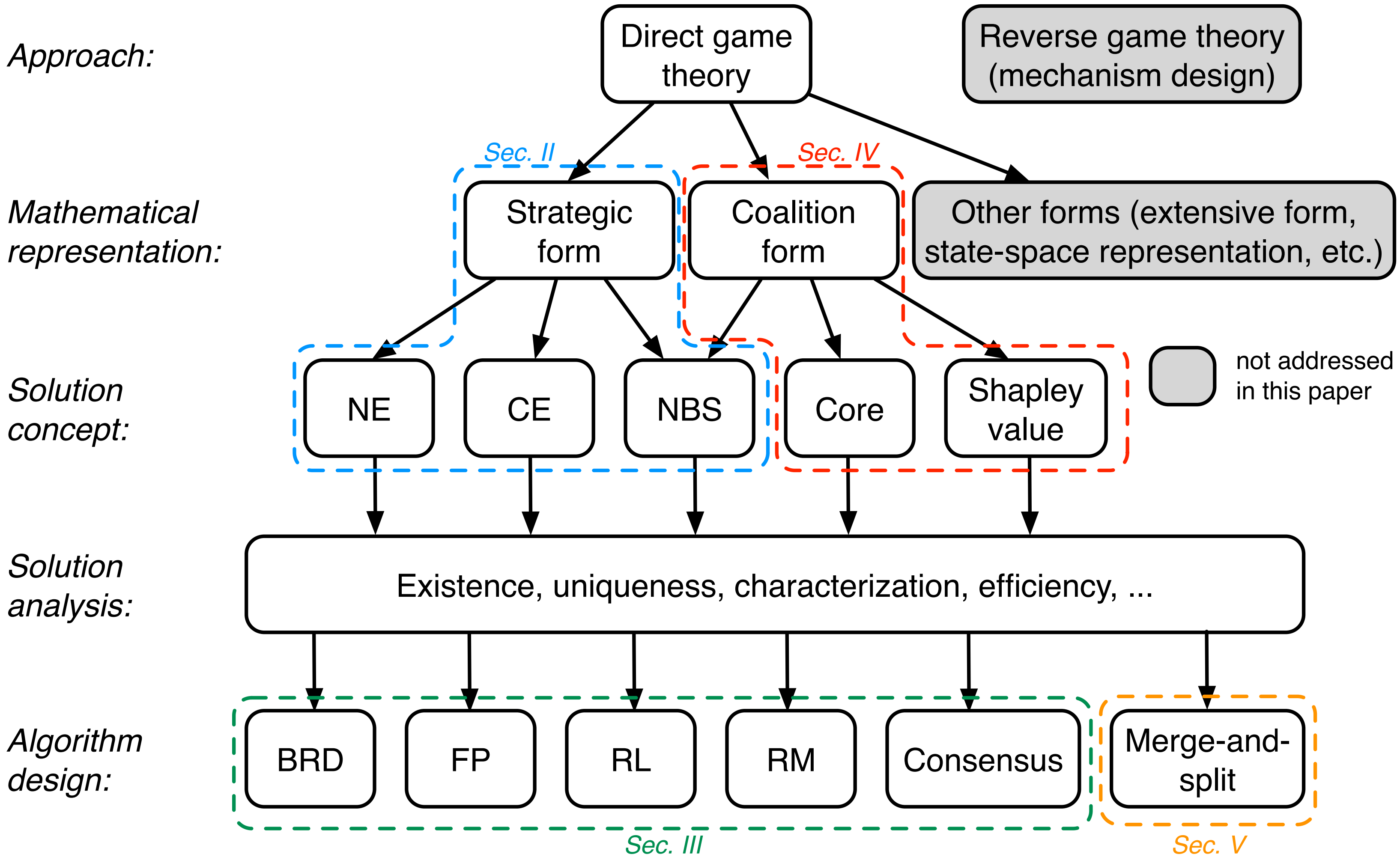
► NE: What is it? Existence? Uniqueness? Efficiency? Existence of a convergent and implementable algorithm?

# What is a game exactly?

## Main mathematical representations

- Strategic or normal form games.
- Extensive form games.
- Coalitional form games.

# Cheap map of the game theory jungle




[Bacci et al 2016]

# 1. Strategic form games

# Strategic form

## Game $\equiv$ triplet:

$$\mathcal{G} = \left(\mathcal{K}, \{\mathcal{S}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}}\right).$$

- $\mathcal{K} = \{1, ..., K\}$ is the set of players.
- $\mathcal{S}_i$ is the set of strategies for player $i$.
- Player $i$'s payoff/utility function:

$$\begin{array}{rccc} u_i : & \mathcal{S}_1 \times ... \times \mathcal{S}_i \times ... \times \mathcal{S}_K & \rightarrow & \mathbb{R} \\ & \underbrace{(s_1, ..., s_i, ... s_K)}_{s \,:\, \text{strategy profile}} & \mapsto & u_i(s_i, s_{-i}) \end{array}.$$

# Remark on the strategic form

$\exists$ a more general form:

$$\mathcal{G} = (\mathcal{K}, \{\mathcal{S}_i\}_{i\in\mathcal{K}}, \succeq_i)$$

# Existence of a utility function (1/2)

**Proposition (Debreu 1954): there exist no utility functions for lexicographic ordering on $\mathbb{R}^2$.**

**Proposition:** there exists a utility function for every transitive and complete ordering on any countable set:

- completeness: $x \succeq y$ or $y \succeq x$ or both;
- transitivity: "$x \succeq y$ and $y \succeq z$" $\Rightarrow x \succeq z$.

**Proposition (Debreu 1954):** there exists a utility function for every transitive, complete, and continuous ordering on a continuous set $\mathcal{X} \subset \mathbb{R}^N$ provided $\mathcal{X}$ is non-empty, closed, and connected:

- continuity: $\mathcal{B}(x) = \{y \in \mathcal{X} : x \succeq y\}$ and $\mathcal{W}(x) = \{y \in \mathcal{X} : y \succeq x\}$ are closed.

**Remark (connectedness):** $\mathcal{X}$ is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, $\mathcal{X}$ is said to be connected.

# Existence of a utility function (2/2)

**Theorem (preferences over lotteries):** the complete and transitive preference ordering $\succeq$ over $\Delta(\mathcal{S})$ admits a utility function (expected utility) if and only if $\succeq$ meets the VNM axioms of independence and continuity:

- VNM independence axiom: $x \succ y \Rightarrow (1-\mu)x + \mu z \succ (1-\mu)y + \mu z, \ \mu \in ]0,1[$;
- VNM continuity axiom: $x \succ y \succ z \Rightarrow \exists \mu \in ]0,1[, \ (1-\mu)x + \mu z \succ y \succ (1-\mu)z + \mu x$.

**Remarks:** the Allais paradox (1953), voting procedures.

# Prisoner's dilemma under strategic form

Market power terminology [Singh 2009]:

- Players $\mathcal{K} = \{\text{G1}, \text{G2}\}$.
- Strategies are merely actions $\mathcal{S}_1 = \mathcal{S}_2 = \{\text{low}, \text{high}\}$.
- Utility function for Player 1:

$$u_1(s_1, s_2) = \left| \begin{array}{lllll} 0 & \text{if} & (s_1, s_2) & = & (\text{low}, \text{high}) \\ 1 & \text{if} & (s_1, s_2) & = & (\text{high}, \text{high}) \\ 3 & \text{if} & (s_1, s_2) & = & (\text{low}, \text{low}) \\ 4 & \text{if} & (s_1, s_2) & = & (\text{high}, \text{low}) \end{array} \right. .$$

## Generator's dilemma under matrix form

| G1, G2 | high price | low price |
|---|---|---|
| high price | (3, 3) | (0, 4) |
| low price | (4, 0) | (1, 1) |

# A fundamental solution concept: The Nash equilibrium (NE)

**Pure Nash equilibrium.** Strategy vector/profile such that

$$\forall i \in \mathcal{K}, \forall s_i \in \mathcal{S}_i,\ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

**Mixed Nash equilibrium** ...

# Mixed strategies

# Mixed strategies

# Mixed strategies and mixed NE

- **Mixed strategies** $\pi_i \in \Delta(\mathcal{S}_i)$ with

$$\Delta(\mathcal{S}_i) = \left\{ x \in \mathbb{R}^{|\mathcal{S}_i|} : x_j \geq 0, \sum_j x_j = 1 \right\}$$

- **Expected utility**

$$\widetilde{u}_i(\pi_1, ..., \pi_K) = \mathbb{E}_{\pi_1 \otimes ... \otimes \pi_K} \left[ u_i(s_1, ..., s_K) \right].$$

- **Mixed Nash equilibrium**

$$\forall i \in \mathcal{K}, \forall \pi_i \in \Delta(\mathcal{S}_i), \; \widetilde{u}_i(\pi_i^*, \pi_{-i}^*) \geq \widetilde{u}_i(\pi_i, \pi_{-i}^*).$$

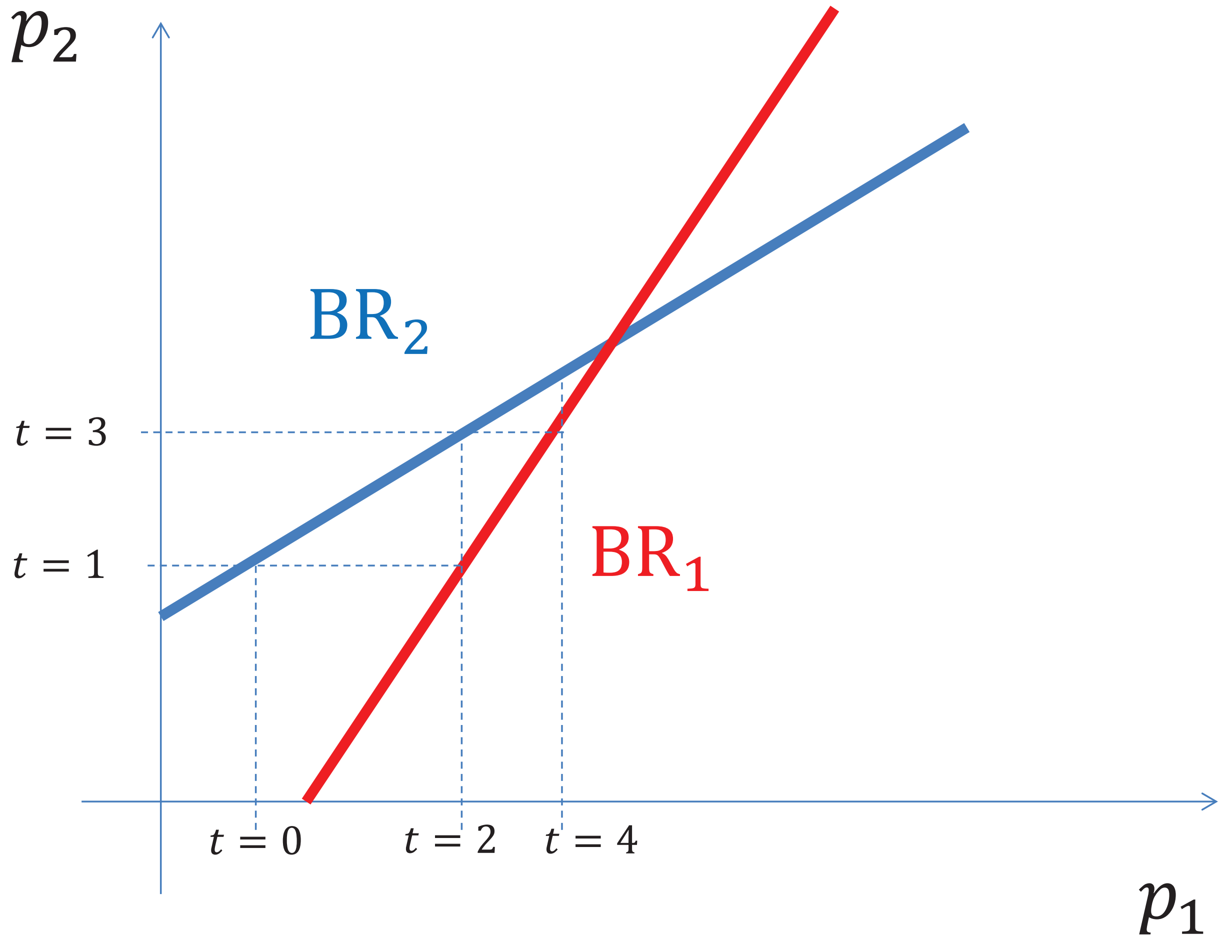
## Three strengths of the Nash equilibrium

- **Stability property (once you are there).**
- **Dynamical property (to get there).**
- **It “always” exists.**

## Best-response

$$\mathrm{BR}_i(s_{-i}) = \arg\max_{s_i \in \mathcal{S}_i} u_i(s_i, s_{-i}).$$

# Illustration for continuous sets [Cournot 1838]



## The sequential best-response dynamics (1/3)

| G1, G2 | high | low |
|---|---|---|
| high | (3, 3) | (0, 4) |
| low | (4, 0) | (1, 1) |

# The sequential best-response dynamics (2/3)

| G1, G2 | high | low |
| --- | --- | --- |
| high | (3, 3) | (0, 4) |
| low | (4, 0) | (1, 1) |

## The sequential best-response dynamics (3/3)

| G1, G2 | high | low |
|---|---|---|
| high | (3, 3) | (0, 4) |
| low | (4, 0) | (1, 1) |

# Nash equilibrium characterization

A strategy profile $s^*$ is an NE of $\mathcal{G}$ iff:

$$s_i^* \in \mathrm{BR}_i(s_{-i}^*) \iff s^* \in \mathrm{BR}(s^*)$$

where

$$\begin{aligned} \mathrm{BR} : \ \mathcal{S} &\rightarrow \mathcal{S} \\ s &\mapsto \mathrm{BR}_1(s_{-1}) \times \mathrm{BR}_2(s_{-2}) \times \ldots \times \mathrm{BR}_K(s_{-K}) \end{aligned}$$

## Nash equilibrium existence (finite games)

**Nash existence theorem [Nash 1950].**
$\mathcal{S} = \mathcal{S}_1 \times ... \times \mathcal{S}_K$ is finite. Then, there is a mixed NE.

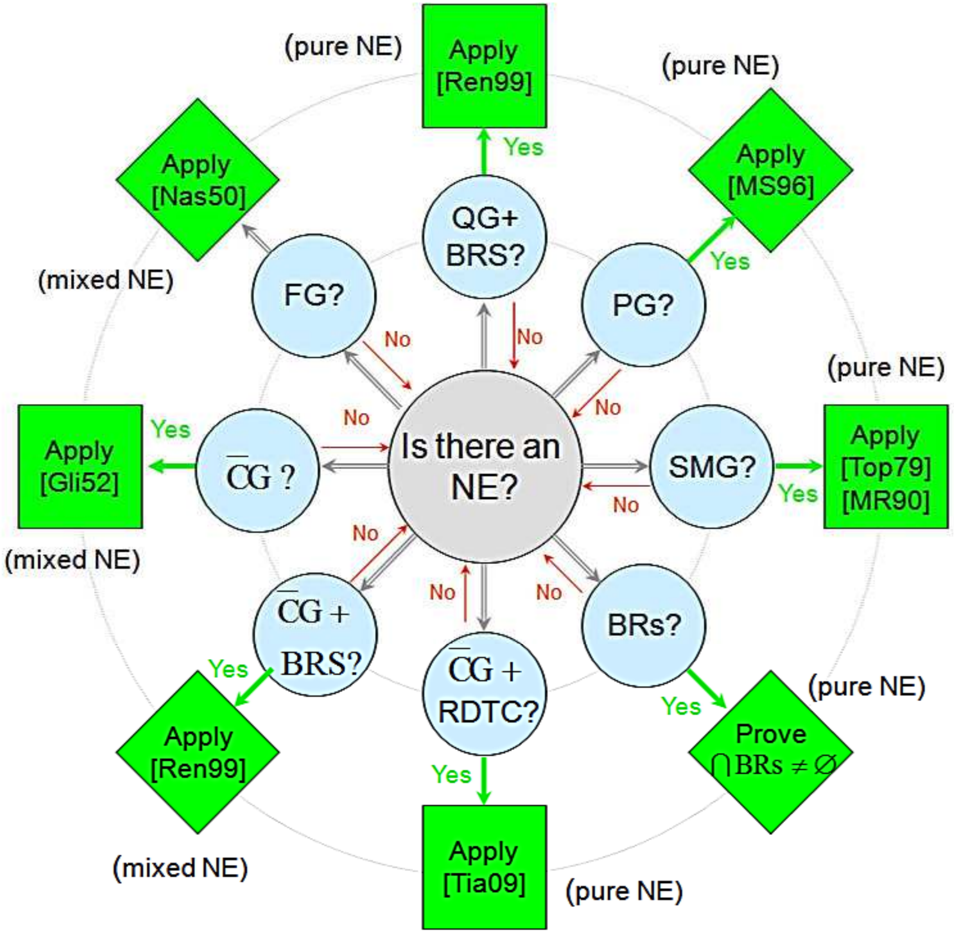
**Kuhn existence theorem [Kuhn 1953].** Every finite game of perfect information has at least one pure NE.

## Nash equilibrium existence (continuous strategy sets)

**Glicksberg theorem [Glicksberg 1952].** $\mathcal{S}_i$ compact, $u_i$ continuous in $s$. Then, there is a mixed NE.

**Debreu-Fan-Glicksberg theorem [Debreu, Fan, Glicksberg 1952].** Above assumptions & $u_i$ quasiconcave in $s_i$. Then, there exists a pure NE.

# More about the existence of NE




[Lasaulce & Tembine 2011]

# Simplified methodology for studying NE

| Static games | Dynamic games |
|---|---|
| Existence | Existence |
| Uniqueness | Utility region characterization ; uniqueness |
| Efficiency | Design of strategies |

# Uniqueness (concave games)

## Rosen theorem [Rosen 1965]

- $\mathcal{S}_i$ compact convex.
- $u_i$ continuous in $s$.
- $u_i$ concave in $s_i$.
- Diagonally strict concavity:

$$\exists r > 0,\ \forall s \neq s',\ \ [s' - s]^T [\gamma_r(s) - \gamma_r(s')] > 0$$

where

$$\gamma_r(s) = \left( r_1 \frac{\partial u_1}{\partial s_1}(s), ..., r_K \frac{\partial u_K}{\partial s_K}(s) \right).$$

Then, there is a unique pure NE.

# Concave game example

**Utility:**

$$u_1(\boldsymbol{A}_1, \boldsymbol{A}_2) = \mathbb{E}\log\left|\boldsymbol{I} + \boldsymbol{X}_1\boldsymbol{A}_1\boldsymbol{X}_1^H + \boldsymbol{X}_2\boldsymbol{A}_2\boldsymbol{X}_2^H\right| - \mathbb{E}\log\left|\boldsymbol{I} + \boldsymbol{X}_2\boldsymbol{A}_2\boldsymbol{X}_2^H\right|$$

**Action space:**

$$\mathcal{A}_1 = \left\{\boldsymbol{A}_1 \geq 0, \boldsymbol{A}_1^H = \boldsymbol{A}_1, \mathrm{Tr}\boldsymbol{A}_1 \leq a\right\}$$

1. DSC is met (trace inequality); 2. NE determination (random matrix theory) [Belmega et al 2011]

# Uniqueness (standard games)

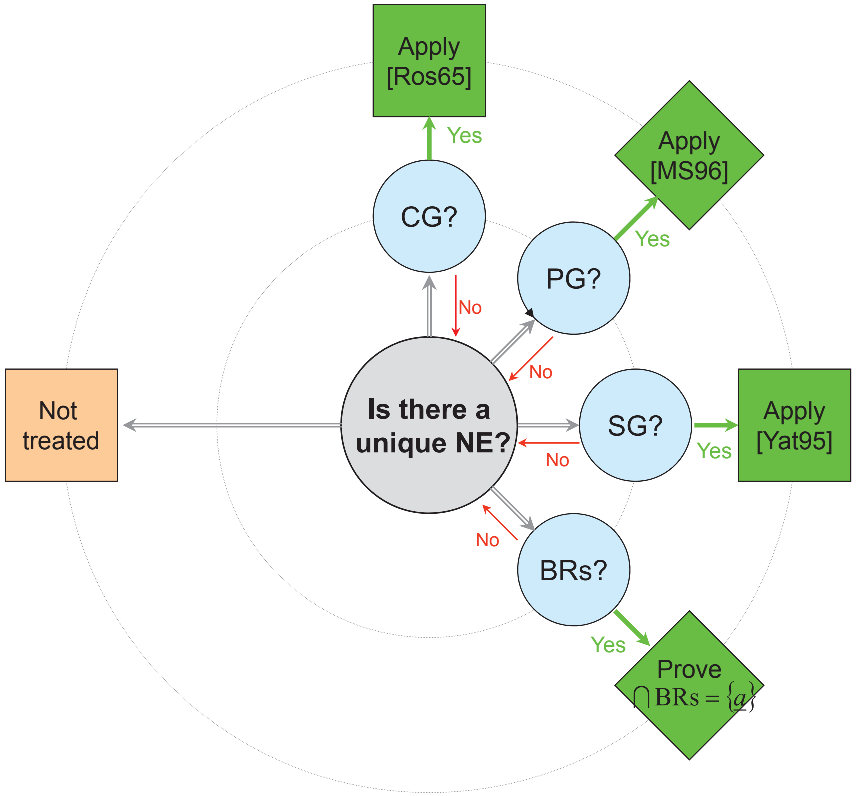
**Definition (standard functions)** A vector function $g : \mathbb{R}_+^K \to \mathbb{R}_+^K$ is standard if we have:

- Monotonicity: $\forall (x, x') \in \mathbb{R}_+^{2K}$, $x \leq x' \Rightarrow g(x) \leq g(x')$.
- Scalability: $\forall \alpha > 1$, $\forall x \in \mathbb{R}_+^K$, $g(\alpha x) < \alpha g(x)$.

**Theorem [Yates 1995]** If $\mathrm{BR} = (\mathrm{BR}_1, ..., \mathrm{BR}_K)$ is standard, then there is a unique pure NE.

**Remark:** BR intersection.

# More about the uniqueness of NE




[Lasaulce & Tembine 2011]

# Obviously the Nash equilibrium has also drawbacks

- Efficiency: typical consequence of partial control
- Correlation: mixed NE assume independent lotteries
- Strategic stability: only stable to single deviations
- Not fully adapted to QoS constraints

For more drawbacks see [Perlaza & Lasaulce 2014]

## Solution concepts for strategic/extensive form games

- **Pure/mixed Nash equilibrium, Wardrop equilibrium**,
- **correlated equilibrium, coarse correlated equilibrium,**
- $N-$ strong equilibrium,
- Nash equilibrium refinements : trembling hand perfect equilibrium, proper equilibrium,
- $\epsilon-$Nash equilibrium,
- logit equilibrium,

## Solution concepts for strategic/extensive form games. Continued

- maxmin strategy profiles,
- Bayesian equilibrium,
- evolutionary stable solution,
- satisfaction equilibrium, generalized Nash equilibrium,
- Stackelberg equilibrium,
- **Pareto optimum, social optimum**,
- bargaining solutions (Nash, egalitarian, Kalai-Smorodinsky, etc.),...

# How to measure efficiency: Pareto efficiency

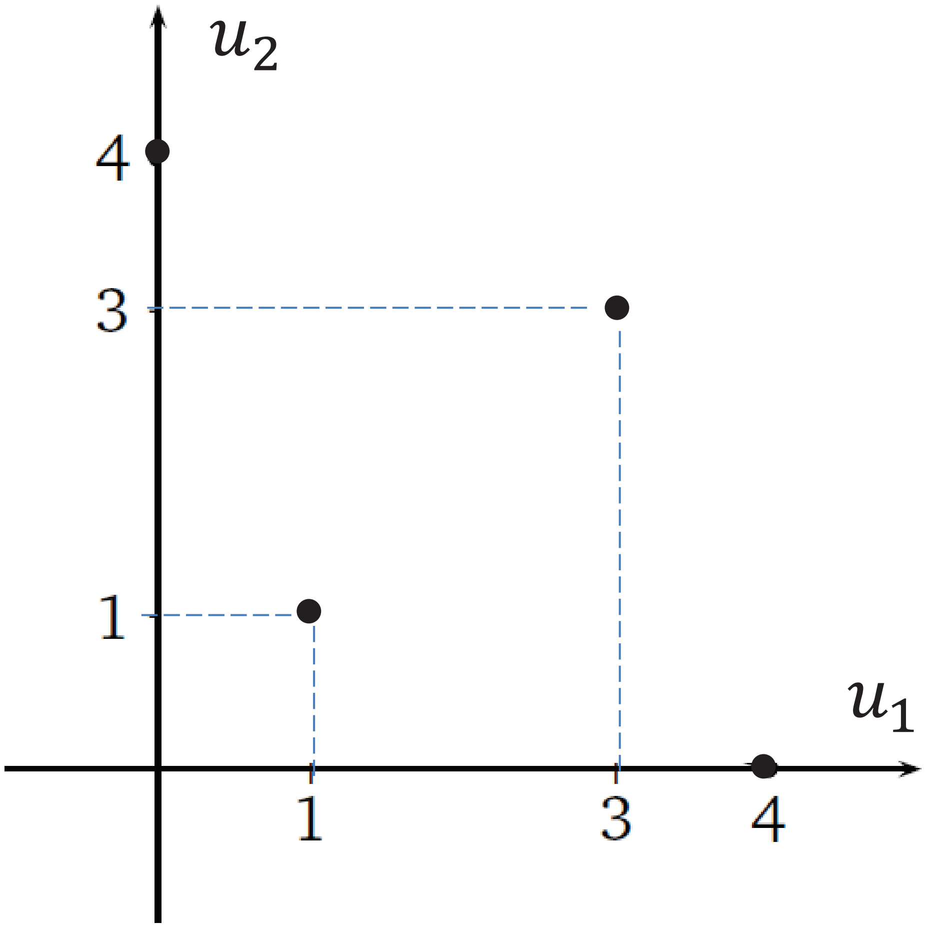
**Definition (Pareto-dominance):** $s$ Pareto-dominates $s'$ if:

$$\forall i \in \mathcal{K},\ u_i(s) \geq u_i(s'),$$

with strict inequality for at least one player.

**Definition (Pareto-optimum):** $s^*$ is Pareto-optimal (-efficient) if it is dominated by no other profile.

# Illustration of Pareto optimality



# How to measure efficiency : Social welfare

**Definition (social welfare):** the social welfare of a game is defined as:

$$w = \sum_{i=1}^{K} u_i.$$

**Definition (social optimum):** an SO is a strategy profile which maximizes $w$.

**Remark:** An SO is a PO.

# How to measure efficiency : PoA and PoS

## Definition (price of anarchy):

$$\mathrm{PoA} = \frac{\max_{s \in \mathcal{S}} w(s)}{\min_{s^* \in \mathcal{S}^{\mathrm{NE}}} w(s^*)}$$

where $\mathcal{S}^{\mathrm{NE}}$ is the set of NE of the game.

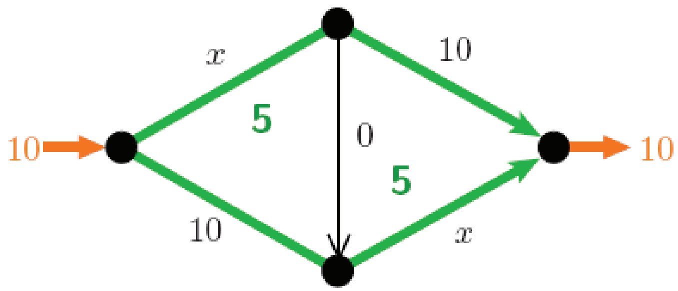
## Definition (price of stability):

$$\mathrm{PoS} = \frac{\max_{s \in \mathcal{S}} w(s)}{\max_{s^* \in \mathcal{S}^{\mathrm{NE}}} w(s^*)}.$$

[Papadimitriou 2001] [Anshelevich et al 2004].

# Example: PoA in non-atomic routing games

The network cost is defined by:




$$C(x) = \sum_{r \in \mathcal{R}} c_r(x_r)x_r$$

**Theorem.** For polynomials costs of maximum degree $d$, the PoA is bounded as:

| degree | 1 | 2 | 3 | 4 | ... | $d$ |
|---|---|---|---|---|---|---|
| PoA | $\frac{4}{3}$ | 1.626 | 1.896 | 2.151 | ... | $\Omega(\frac{d}{\ln(d)})$ |

[Correa et al 2005].

## How to improve efficiency

### Possible approaches (non-exhaustive list)

▶ Introduce pricing.

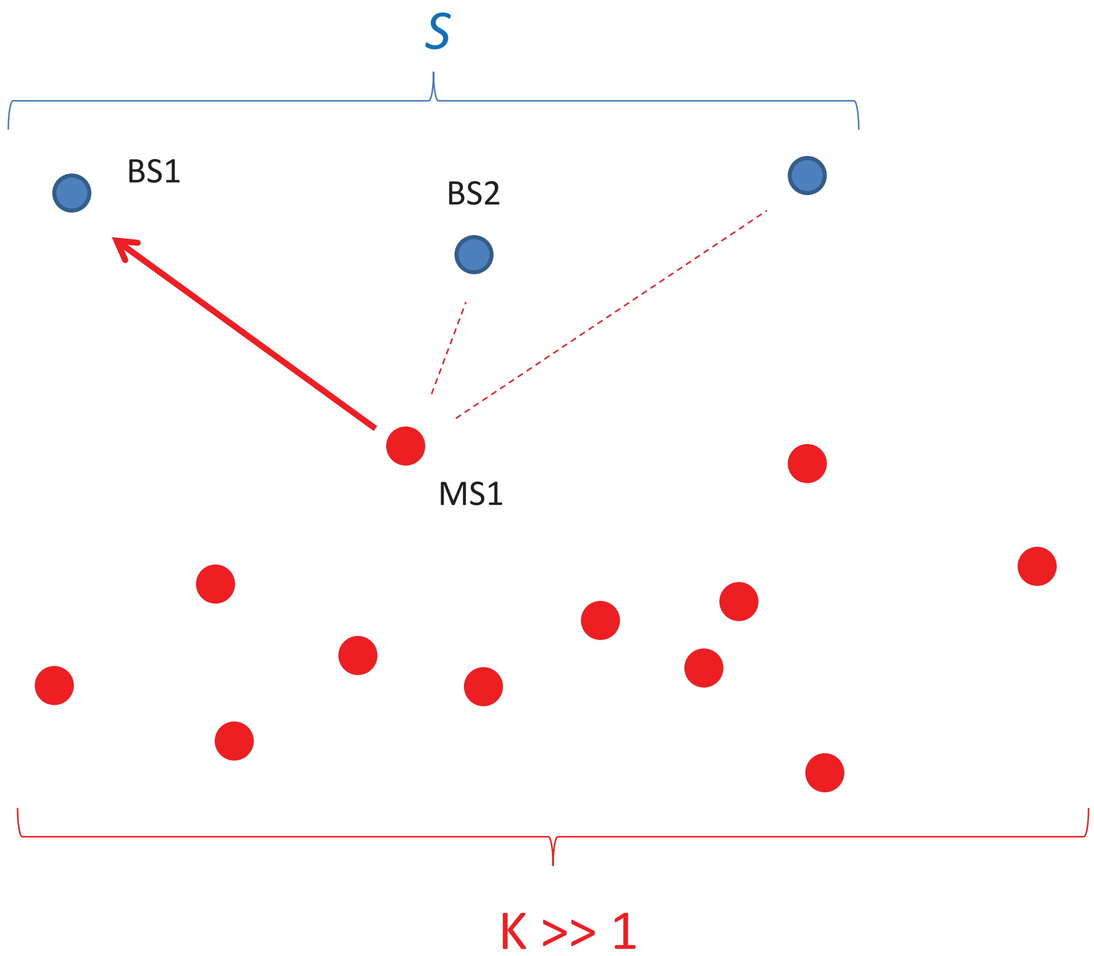
▶ Introduce hierarchy.

▶ Introduce coordination (e.g., correlated equilibrium).

▶ Introduce cooperation (bargaining, cooperation plan in dynamic games, agreement/contract in coalitional games, ...).

# How to improve efficiency (pricing). Example

## Scenario



# Example (utilities)

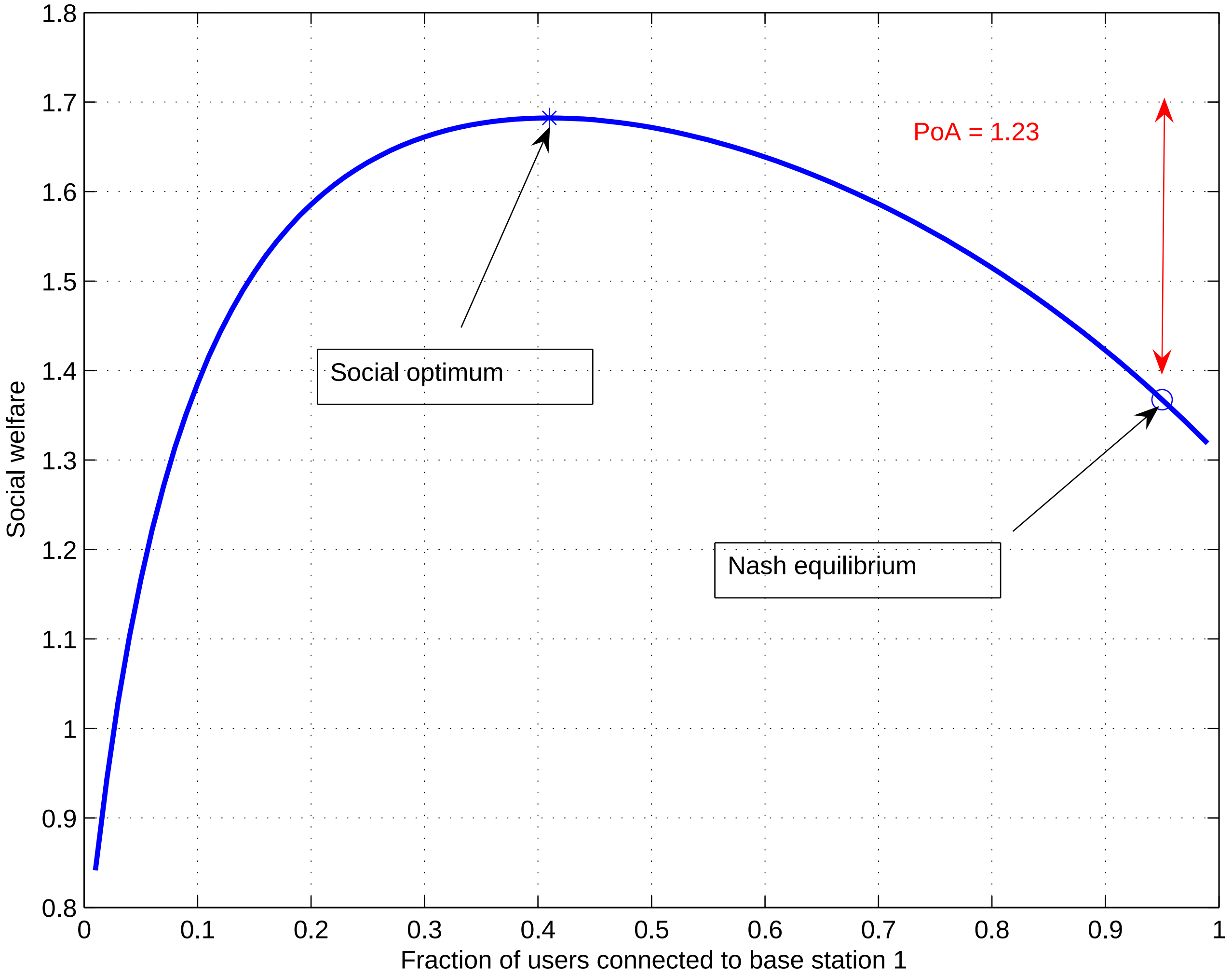
**Utility of player $k$ when connecting to base station $s$**

$$v_{k,s}(x) = \log\left[1 + \frac{1}{a_s + bx_s}\right],$$

$a_s > 0, b > 0$.

# Example (illustration)

## Social welfare for $S = 2$



# Example (pricing and modified game)

Let $n_k$ be the data volume to be transferred:

$$\tau_{k,s}(x) = \frac{n_k}{\upsilon_{k,s}(x)}.$$

Cost function of the new game:

$$c_{k,s}(x) = p(\tau_{k,s}(x)) + \beta_s.$$

Parameter adjustment $\rightarrow$ desired solution.

## How to improve efficiency: introduce coordination through correlated equilibria

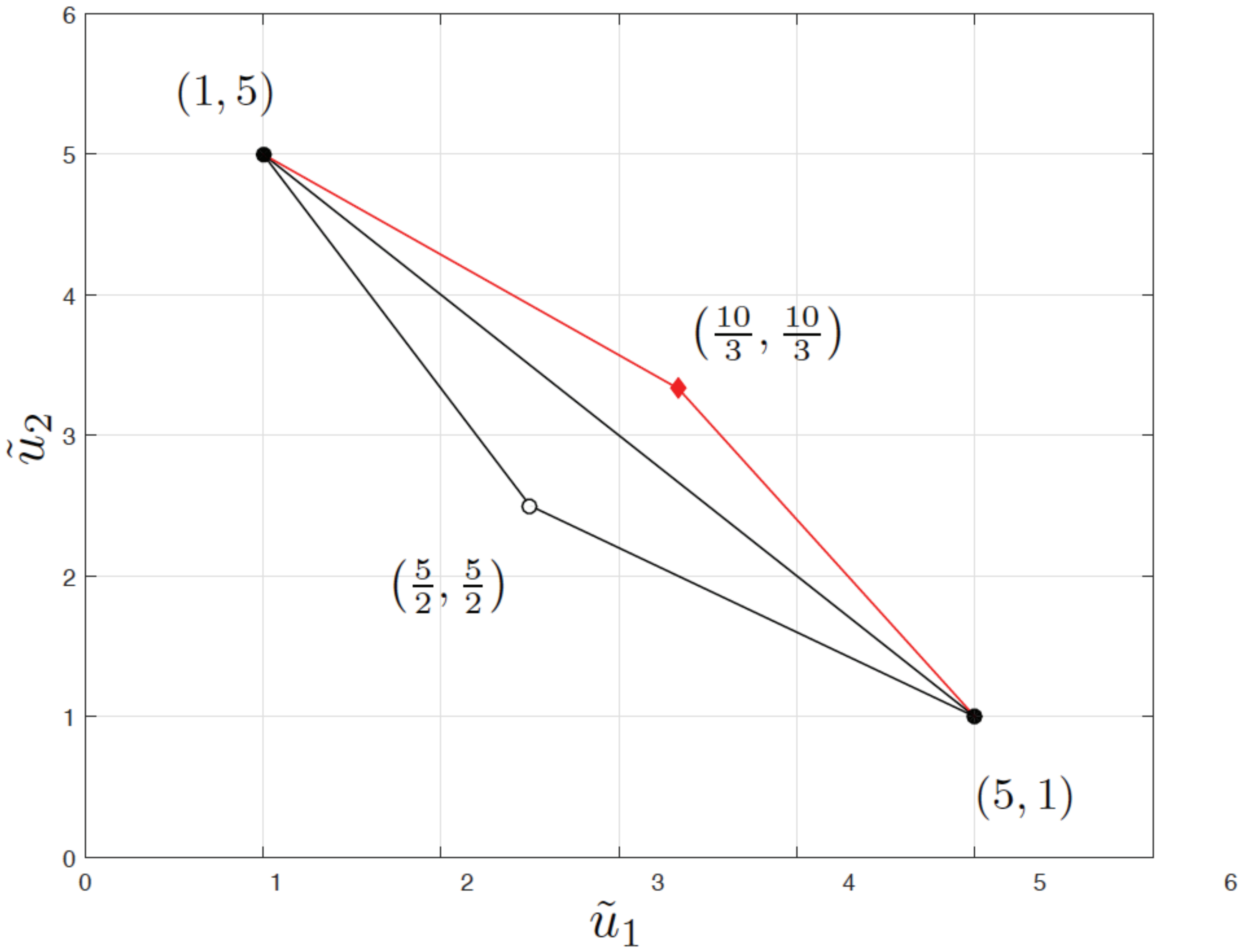
**Definition (correlated equilibrium)** Let $\sigma_k : \mathcal{A}_k \rightarrow \mathcal{A}_k$ be a mapping. Then $q^{\mathrm{CE}}$ is a CE if

$$\forall k, \forall \sigma_k,$$
$$\sum_{a \in \mathcal{A}} q^{\mathrm{CE}}(a_k, a_{-k}) u_k(a_k, a_{-k}) \geq \sum_{a \in \mathcal{A}} q^{\mathrm{CE}}(a_k, a_{-k}) u_k(\sigma_k(a_k), a_{-k}),$$

**Example (CR coordination game)**

| | Low | High |
|---|---|---|
| High | (5 , 1 ) | (0, 0) |
| Low | (4, 4 ) | (1 , 5 ) |

# Set of correlated equilibria



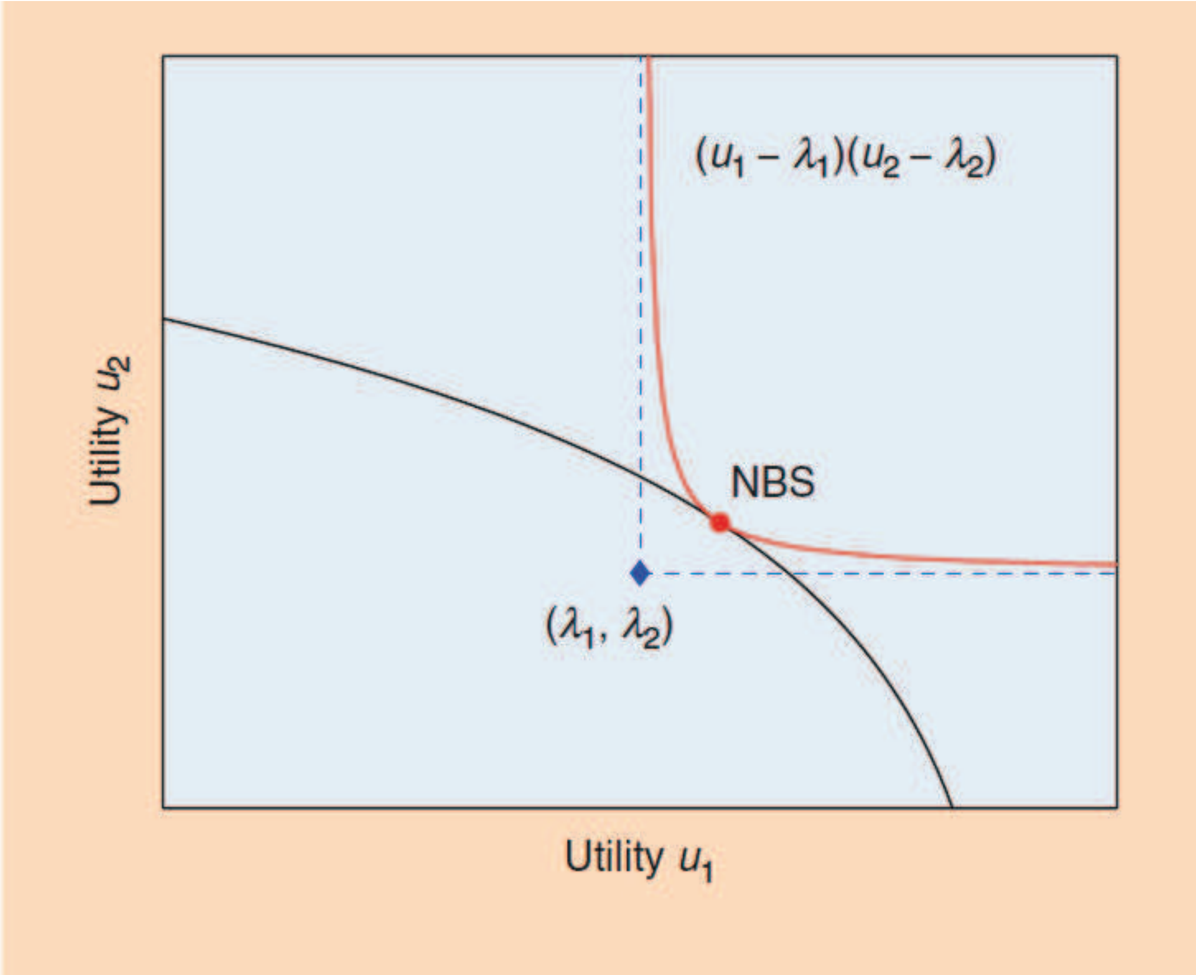
# Nash bargaining solution

**Definition** The NBS is the unique solution of

$$\max_{(u_1,u_2)\in\mathcal{U}} \quad (u_1-\lambda_1)(u_2-\lambda_2)$$
$$\text{subject to} \quad u_1 \geq \lambda_1,\ u_2 \geq \lambda_2$$

where $\mathcal{U}$ is the game feasible utility set.

# Illustration of the NBS




[FIG7] The graphical interpretation of the NBS point (red circle) as the intersection between the Pareto boundary of $\mathcal{U}$ and the hyperbola $(u_1 - \lambda_1)(u_2 - \lambda_2) = \kappa$, where the status quo $\lambda = (\lambda_1, \lambda_2)$ is represented by the blue diamond.

# 2. Dynamic games

# Dynamic games

## Typical ingredients

- Several stages.

- Notions of game history, action plans.

- Average/long-term utility.

- The stage utility is state-dependent ($u_i(a, x)$).

**Informal definition.** A game in which at least one player can use a strategy depending on previously played actions. No universal definition, only special classes.

# Important classes of dynamic games

- **Repeated games ($\emptyset$).**
- **Stochastic games (MDP).**
- **Differential/difference games (OC).**
- Mean-field games.
- Evolutionary games.

# Repeated games with perfect monitoring

**Definition (game history):** $\forall t \geq 1$,
$h_t = (a(1), ..., a(t-1)) \in \mathcal{H}_t$ where $\mathcal{H}_t = \mathcal{A}^{t-1}$.

**Definition (pure strategy):** A pure strategy for player $i \in \mathcal{K}$ is a sequence $(\tau_{i,t})_{t\geq 1}$ with

$$\begin{array}{rccc} \tau_{i,t}: & \mathcal{H}_t & \to & \mathcal{A}_i \\ & h_t & \mapsto & a_i(t) \end{array}$$

**Definition (behavior strategy):** A behavior strategy for player $i \in \mathcal{K}$ is a sequence $(\widetilde{\tau}_{i,t})_{t\geq 1}$ with

$$\begin{array}{rccc} \widetilde{\tau}_{i,t}: & \mathcal{H}_t & \to & \Delta(\mathcal{A}_i) \\ & h_t & \mapsto & \pi_i(t). \end{array}$$

# Repeated games utilities

**Finitely repeated games.** Let $\tau = (\tau_1, ..., \tau_K)$ and $T \geq 1$:

$$v_i^T(\tau) = \frac{1}{T}\sum_{t=1}^{T} u_i(a(t)).$$

**Infinitely repeated games:**

$$v_i^\infty(\tau) = \lim_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(a(t)).$$

**Discounted repeated games.** Let $0 < \lambda < 1$ be the discount factor:

$$v_i^\lambda(\tau) = \sum_{t=1}^{+\infty} \lambda(1-\lambda)^{t-1} u_i(a(t)).$$

# Equilibria in repeated games

**Definition (equilibrium strategies).** A joint strategy $\tau^*$ supports an equilibrium of the repeated game $\big(\mathcal{K}, \{\mathcal{T}_i\}_{i\in\mathcal{K}}, \{v_i^y\}_{i\in\mathcal{K}}\big)$, $y \in \{T, \infty, \lambda\}$, if:

$$\forall i \in \mathcal{K}, \forall \tau_i',\ v_i^y(\tau^*) \geq v_i^y(\tau_i', \tau_{-i}^*).$$

**Remark (equilibrium analysis):** Existence for finite games, compact games, static games with a Nash equilibrium. In contrast with static games, there can be many equilibria.

# Equilibrium characterization for discounted repeated games with perfect monitoring

**Folk theorem.** The set of equilibrium utilities when $\lambda \to 0$ is given by

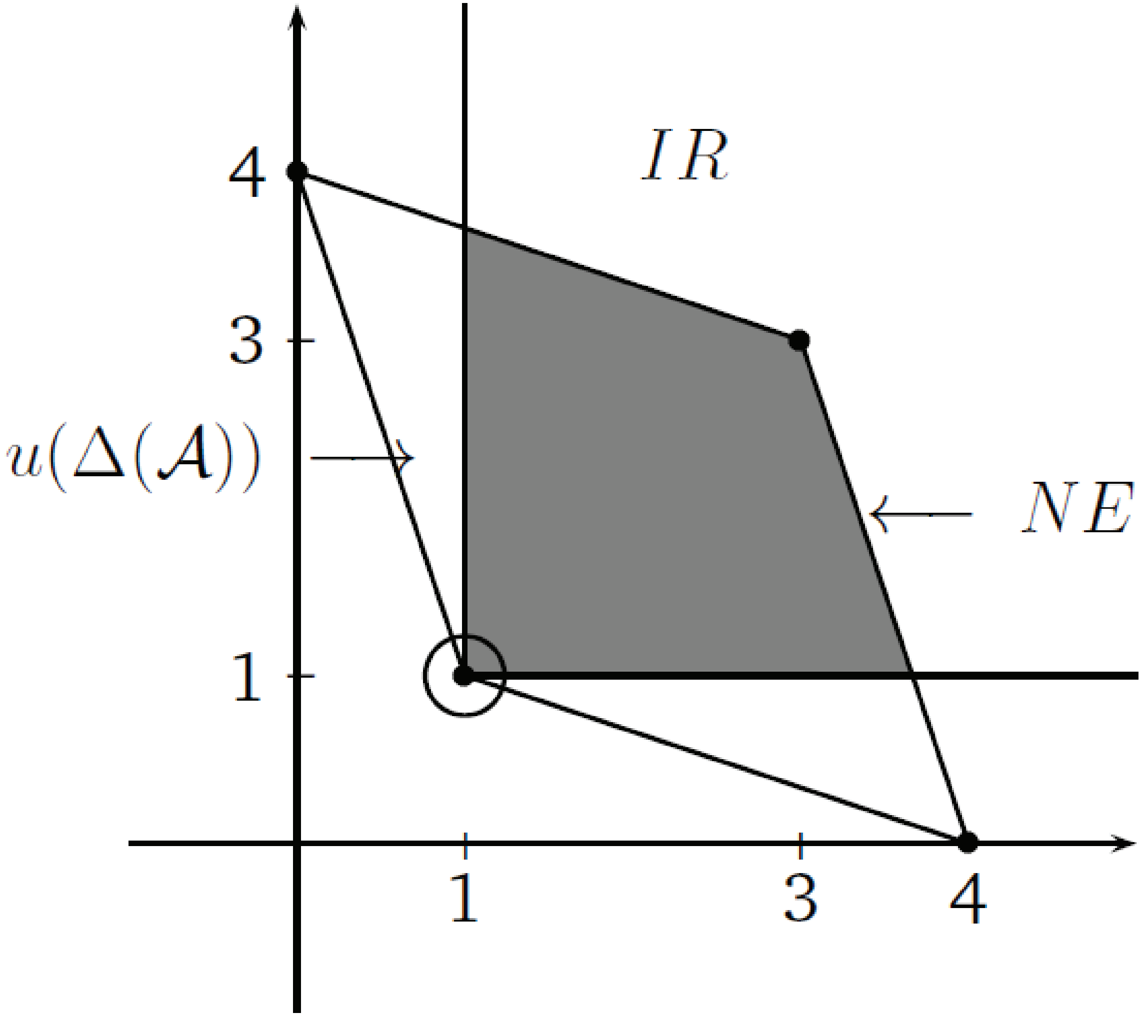
$$E_0 = \mathrm{IR}(\mathcal{G}) \cap \mathrm{co}\left(\mathcal{U}(\mathcal{G})\right)$$

where:

- $\mathrm{IR}(\mathcal{G}) = \{u \in \mathbb{R}^K, \forall i \in \mathcal{K}, u_i \geq \min\limits_{\pi_{-i}} \max\limits_{\pi_i} \widetilde{u}_i(\pi)\}$;
- $\mathcal{U}(\mathcal{G}) = \{u' \in \mathbb{R}^K : \exists a, u(a) = u'\}$.

# Illustration

## Repeated prisoner's dilemma



## Relaxing the perfect monitoring assumption: $2-$connected graphs

**Definition (strongly connected graph)** A graph $\Gamma$ is said to be strongly connected if for each pair of vertices $(i, j)$, there is a directed path from $i$ to $j$.

**Definition ($2-$connected graph)** The graph $\Gamma$ is $2-$connected if, for any vertex $i$, $\Gamma\backslash\{i\}$ is strongly connected.

**Theorem** The following two assertions are equivalent:

**(i)** the observation graph of the infinitely repeated games is $2-$connected;

**(ii)** $E_\infty = \mathrm{IR}(\mathcal{G}) \cap \mathrm{co}\,(\mathcal{U}(\mathcal{G}))$.

[Renault and Tomala 1998]

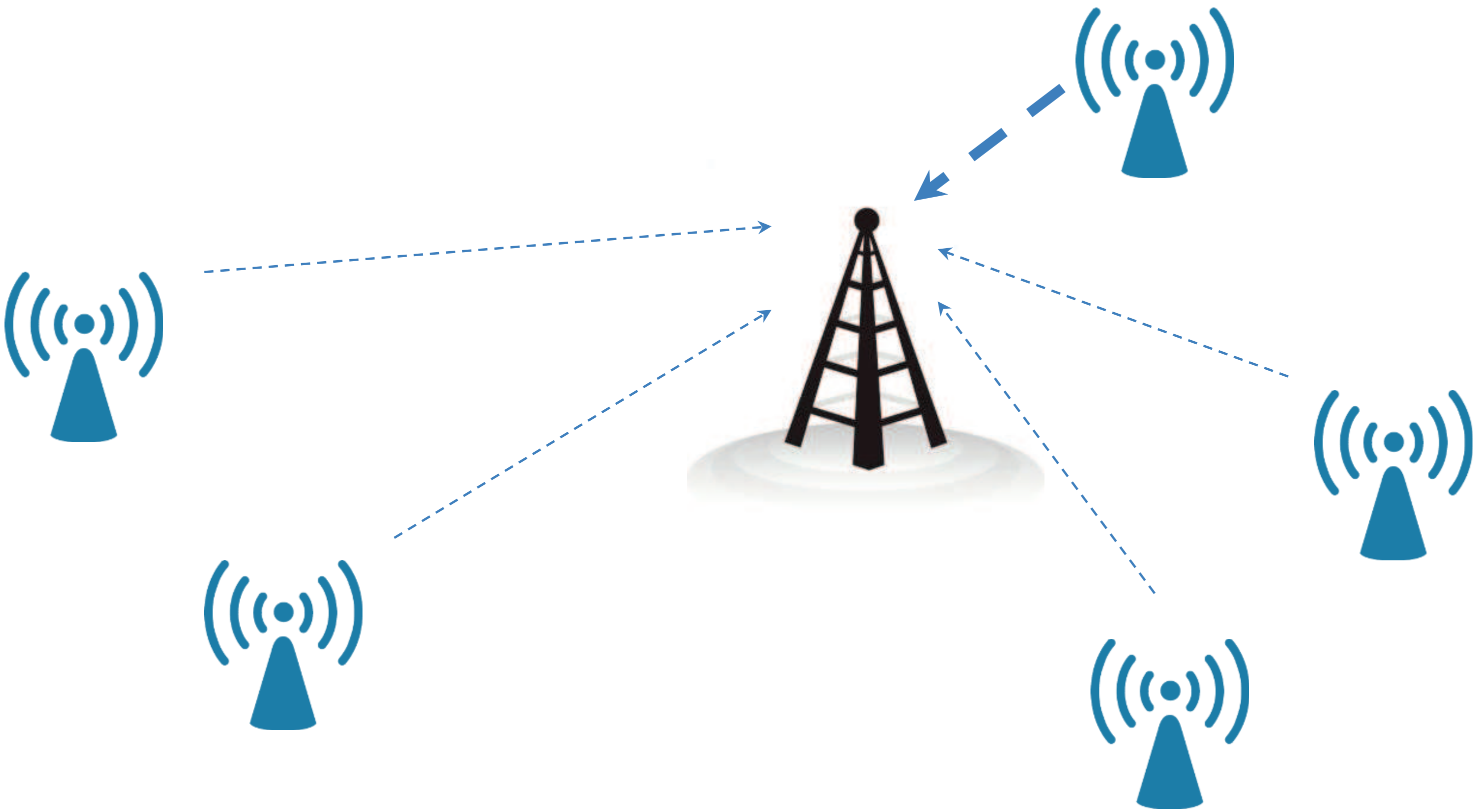
## Case study $\#1$: Wireless power control

- Static game formulation.

- A repeated game formulation.

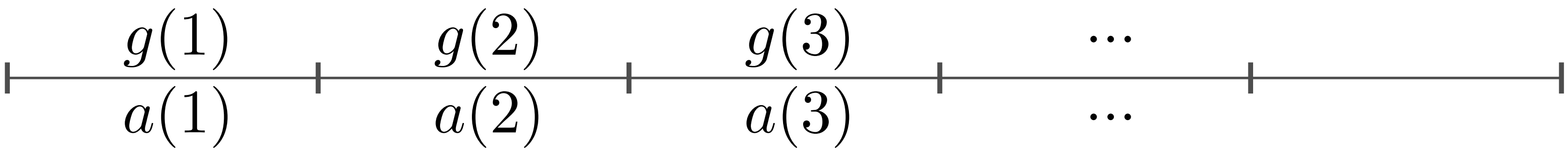
# Near far effect

# A wireless power control scenario



# Modeling the problem as a static game [Goodman & Mandayam 2000]

## Time slots



# Modeling the problem as a static game. Continued

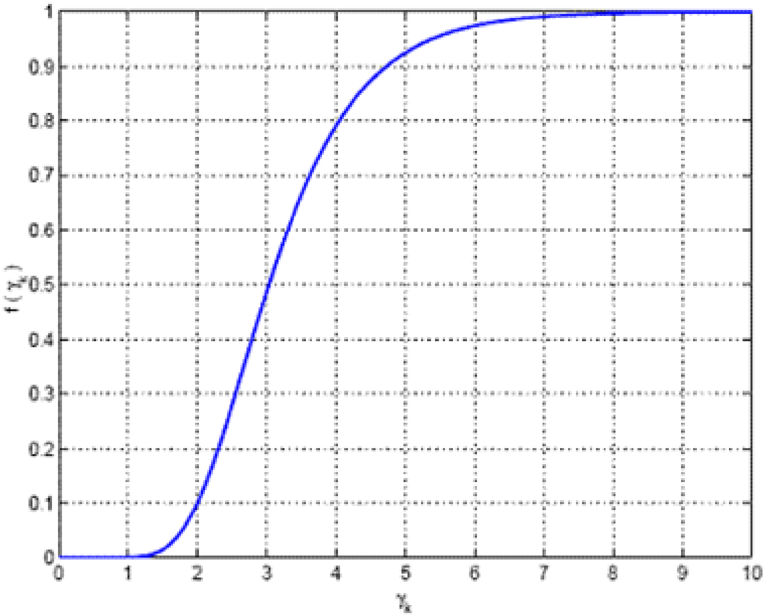
- **Set of players** : $\mathcal{K} = \{1, ..., K\}$.
- **Set of actions** : $\mathcal{A}_i = [0, A^{\max}]$.
- **Utilities** : energy-efficiency;

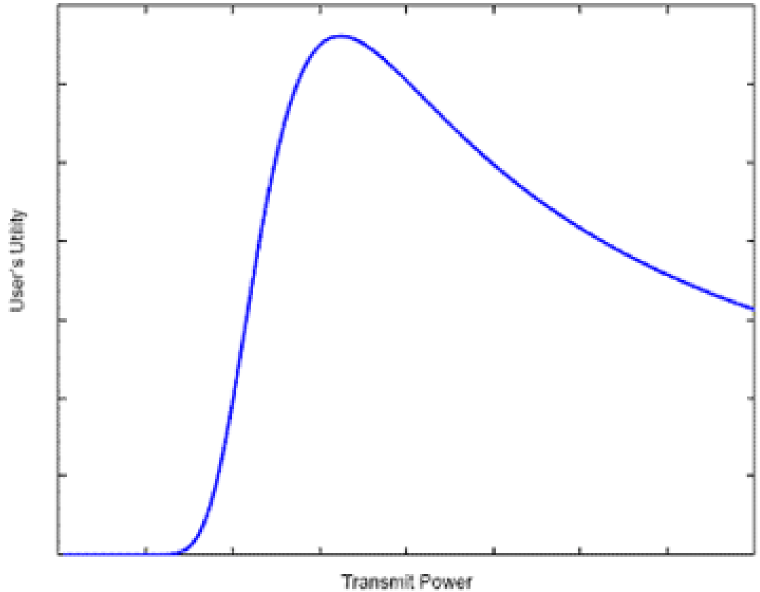
$$u_i(a_i, a_{-i}) = \frac{\text{benefit}}{\text{cost}} = \frac{f(\beta_i)}{a_i}$$

where

$$\beta_i = \frac{g_i a_i}{1 + \sum_{j \neq i} g_j a_j}.$$

# Properties assumed for $f$




- $f$ non-negative, continuous, and non-decreasing.
- $f$ sigmoidal.
- $\lim_{x \to 0} \frac{f(x)}{x} = 0$, $\lim_{x \to +\infty} f(x) = const \leq 1$, $0 \leq f(x) \leq 1$.

# Nash equilibrium analysis (1/3)

## Existence

# Nash equilibrium analysis (1/3)

## Existence

- $\mathcal{A}_i = [0, A_i^{\max}]$: compact, convex.
- $u_i$ is continuous w.r.t. $a = (a_1, ..., a_K)$.
- $u_i$ is quasi-concave w.r.t. $a_i$ ($f(x)$ sigmoidal $\Rightarrow \frac{f(x)}{x}$ is quasi-concave).

# Nash equilibrium analysis (2/3)

## Uniqueness

# Nash equilibrium analysis (2/3)

## Uniqueness

The best response is a function and

$$\forall i \in \mathcal{K},\ \mathrm{BR}_i(a_{-i}) = \frac{\beta}{g_i}\left(1 + \sum_{j \neq i} g_j a_j\right)$$

with $\beta^\star f'(\beta) = f(\beta)$.

The game is standard:

- Monotonicity: $a' \leq a \ \Rightarrow\ \mathrm{BR}(a') \leq \mathrm{BR}(a)$.
- Scalability: $\forall \alpha > 1,\ \mathrm{BR}(\alpha a) < \alpha \mathrm{BR}(a)$.

# Nash equilibrium analysis (3/3)

## Determination (interior point)

Solve the system of equations $\frac{\partial u_i}{\partial a_i}(a) = 0$, which leads to:

$$\forall i \in \{1, ..., K\}, \; a_i^\star = \frac{1}{g_i} \frac{\beta}{1-(K-1)\beta}.$$

**Problem** Generally inefficient solution. How to improve efficiency?

# Introduce pricing

## Main points

- New utility:

$$\widetilde{u}_i(a) = u_i(a) - \alpha a_i, \quad \alpha \geq 0.$$

- Good news. The new NE profile Pareto-dominates $a^\star$.

- Bad news. Uniqueness not guaranteed, convergence under some specific assumption. Global state information is required.

[Saraydar et al 2002].

## Strategic form

$$\mathcal{G}^m = \left(\mathcal{K}, \{\mathcal{T}_i\}_i, \{v_i^m\}_i\right) \quad \text{with} \quad m \in \{T, \lambda\}.$$

If $m = T$:

$$v_i^T = \frac{1}{T} \sum_{t=1}^{T} u_i(\underline{a}(t)).$$

If $m = \lambda \in (0, 1]$:

$$v_i^\lambda = \sum_{t=1}^{+\infty} \lambda(1-\lambda)^{t-1} u_i(\underline{a}(t)).$$

[Le Treust and Lasaulce 2010]

## Public signal choice

$$\omega(t) \triangleq 1 + \sum_{i=1}^{K} g_i a_i(t) = a_i(t) g_i \times \frac{\beta_i(t) + 1}{\beta_i(t)}$$

# Strategic form. Continued

## Pure strategies

$$\begin{array}{rcl} \tau_{i,t} : \quad (\mathcal{A}_i \times \Omega)^{t-1} & \rightarrow & [0, A_i^{\max}] \\ (a_i^{t-1}, \omega^{t-1}) & \mapsto & a_i(t) \end{array}$$

where

- $a_i^{t-1} = (a_i(1), a_i(2), ..., a_i(t-1));$
- $\omega^{t-1} = (\omega(1), \omega(2), ..., \omega(t-1));$
- $\Omega = \left[1, 1 + \sum_{i=1}^{K} g_i^{\max} A_i^{\max}\right].$

# An interesting Nash equilibrium of $\mathcal{G}^m$, $m = T$

## Proposed equilibrium point

$$\tau_{i,t}^{*} = \left| \begin{array}{lll} a_i^{\mathrm{OP}} & \text{if } t \in \{1, 2, \ldots, T - t_0\} \text{ and} & \omega(t) = \frac{1-\gamma}{1-(K-1)\gamma} \\ a_i^{\star} & \text{if } t \in \{T - t_0 + 1, \ldots, T\} \text{ and} & \omega(t) = \frac{1-\gamma}{1-(K-1)\gamma} \\ A_i^{\max} & \text{if} & \omega(t) \neq \frac{1-\gamma}{1-(K-1)\gamma} \end{array} \right.$$

where $\gamma[1 - (K - 1)\gamma]f'(\gamma) - f(\gamma) = 0$ and

$$\forall i \in \mathcal{K}, \quad p_i^{\mathrm{OP}} = \frac{1}{g_i} \frac{\gamma}{1 - (K - 1)\gamma}.$$

# Comments

- To obtain OP, impose $g_j a_j =$ const.
- $t_0$ comes from the equilibrium condition:

$$\text{Let } t_0 = \left\lceil \frac{\frac{f(\alpha)}{\alpha} - \frac{f(\beta)[1-(K-1)\beta]}{\beta}}{\frac{f(\alpha)[1-(K-1)\alpha]}{\alpha} - \frac{f(\alpha)}{\alpha\left(1+\sum_{j\neq i} g_j P_j^{\max}\right)}} \right\rceil$$

- Local knowledge. Pareto domination of the NE of $\mathcal{G}$. Good

in terms of social welfare.

# Time-varying parameter case

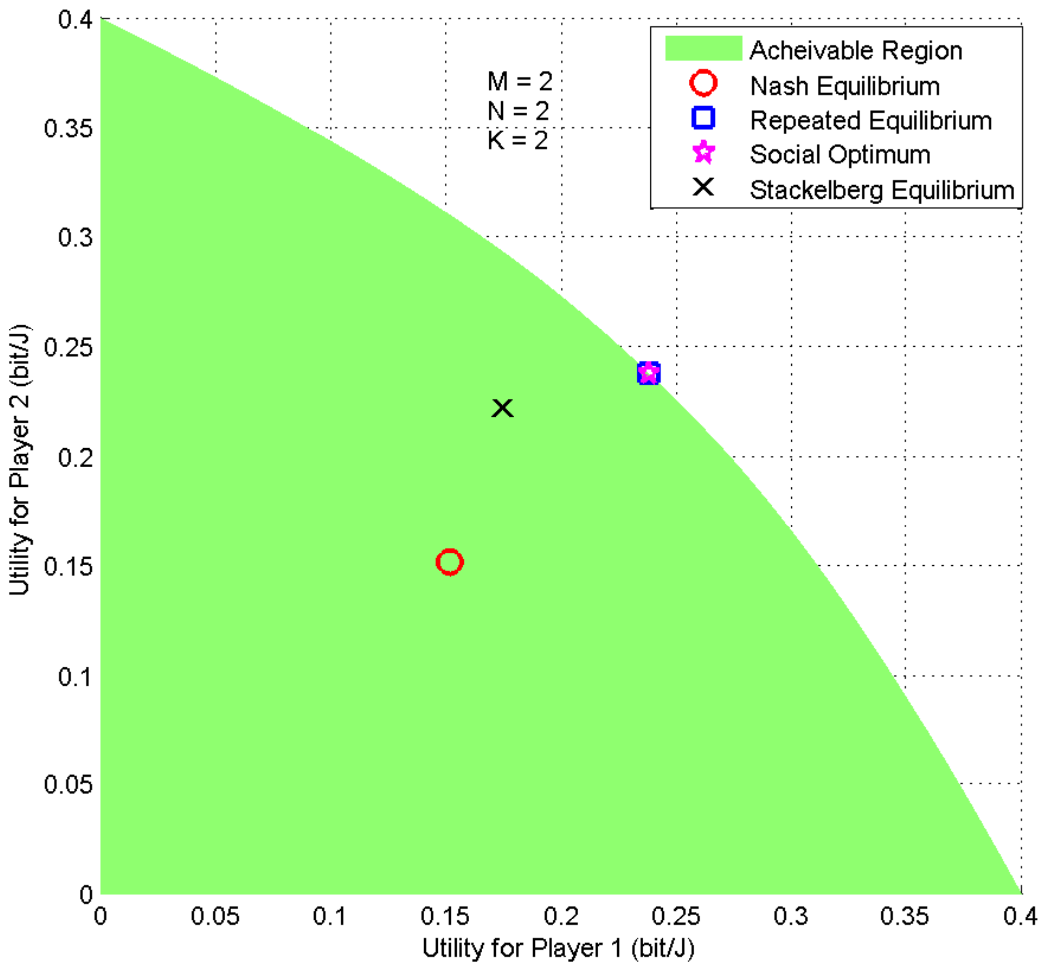
▶ Repeated game methodology holds (worst-case scenario). For instance, $t_0$ becomes:

$$t_0 = \left\lceil \frac{\frac{g_i^{\max}}{g_i^{\min}} \frac{f(\alpha)}{\alpha} - \frac{f(\beta)[1-(K-1)\beta]}{\beta}}{\frac{f(\alpha)[1-(K-1)\alpha]}{\alpha} - \frac{\frac{g_i^{\max}}{g_i^{\min}} f(\alpha)}{\alpha\left(\sum_{j\neq i} P_j^{\max} g_j^{\min} + 1\right)}} \right\rceil .$$

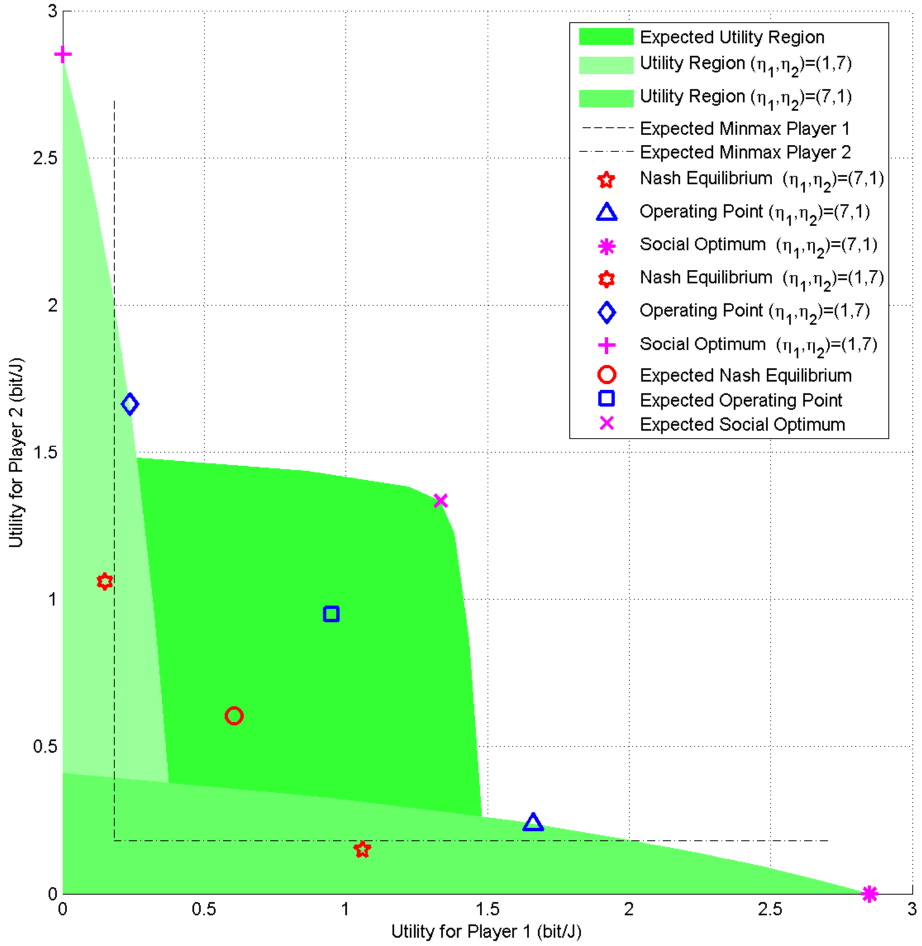
▶ **Stochastic game formulation:** i.i.d. state, $\overline{v}_i^T = \mathbb{E}_{\underline{g}}\left[v_i^T(.)\right]$

- good: better performance;
- bad: more information is needed (parameter distribution).

# Illustration (fixed parameter)



# Illustration (time-varying parameter)




[Mériaux et al 2011]

# Observations

► Stochastic game case = most general case + most efficient policies.

► Importance of characterizing equilibrium points.

# Reminders

# Feasible set characterization for stochastic games with i.i.d. common state

- Stage utilities: $u_i(a_0, a_1, ..., a_K)$; $a_i \in \mathcal{A}_i$, $|\mathcal{A}_i| < \infty$
- Observation/signal structure: $\daleth(s_i|a_0)$, $\Gamma(y_i|a_0, a_1, ..., a_K)$; $|\mathcal{S}_i| < \infty$, $|\mathcal{Y}_i| < \infty$
- Long-term utilities:

$$v_i^{\infty}(\tau_1, ..., \tau_K) = \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_i(A_0(t), A_1(t), ..., A_K(t))\right]$$

*References: [Larrousse and Lasaulce 2013][Larrousse et al 2015][Larrousse et al 2018]*

# Stage game description (example)

- Decision-makers: $\{1, 2\}$; $0 \equiv$ nature.

- Action sets: $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{0, 1\}$.

- Stage utility function:

$$u(a_0, a_1, a_2) = \left| \begin{array}{ll} 1 & \text{if } a_0 = a_1 = a_2 \\ 0 & \text{otherwise} \end{array} \right. .$$

# Observation structure

► Stages: $t \in \{1, 2, ..., T\}$, $T \geq 2$.

► DM 1 knows $a_0^T = (a_0(1), a_0(2), ..., a_0(T))$ and has perfect recall.

► $\forall t \geq 2$, DM 2 knows $a_0^t = (a_0(1), ..., a_0(t-1))$, perfectly monitors DM 1's actions $a_1^t = (a_1(1), ..., a_1(t-1))$, and has perfect recall.

Question: To what extent can they coordinate?

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u(A_0(t), A_1(t), A_2(t))\right] = \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u_t\right]$$

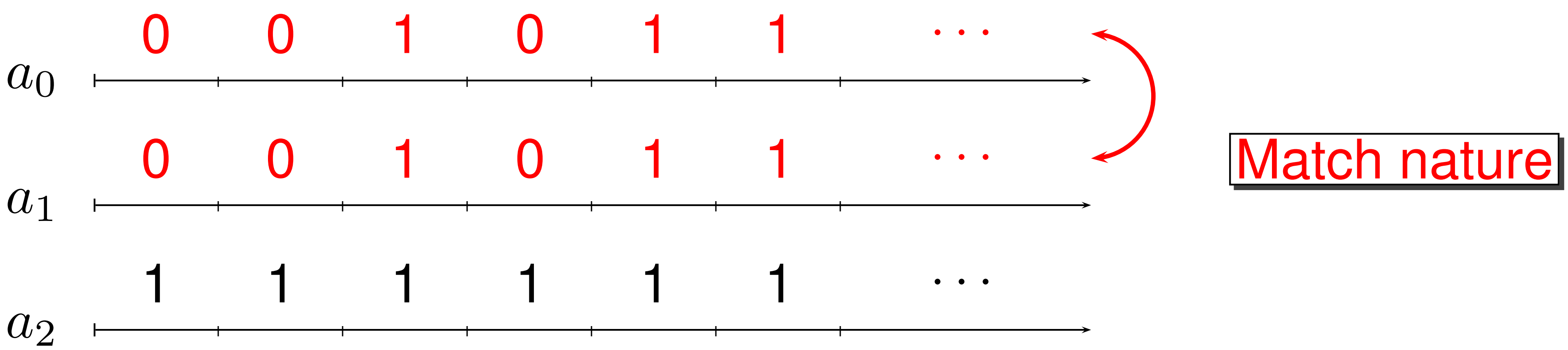
# Trivial upper bound

## Centralized case

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u_t\right] \leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} \max_{(a_1,a_2)} u(a_0(t), a_1, a_2)\right] = 1.$$

# Average utility

- Scheme 1:

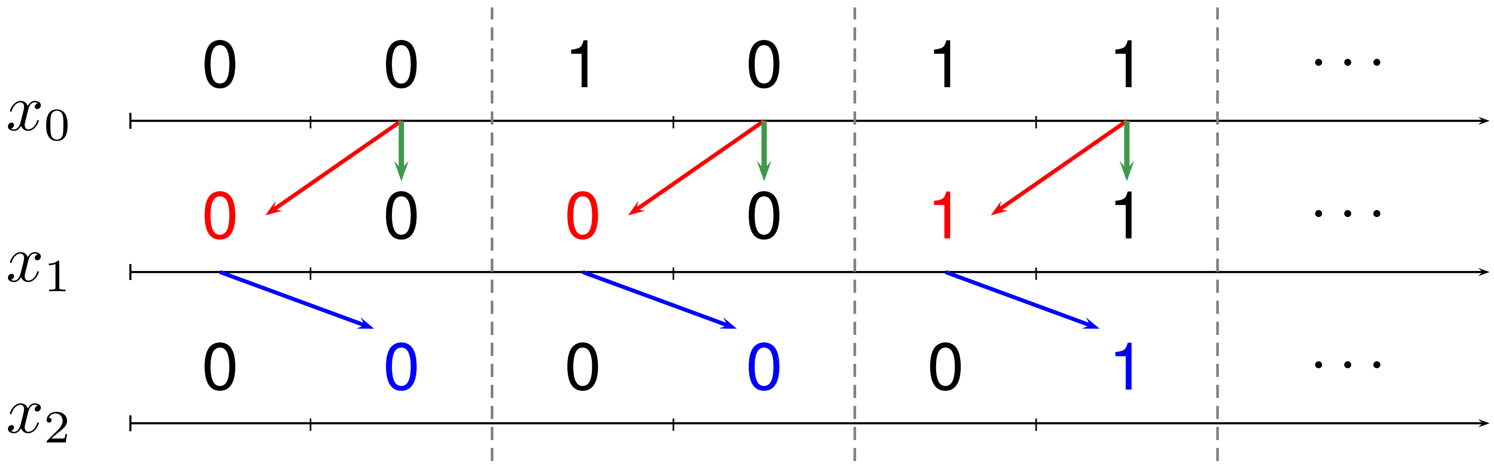



- Average utility

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u_t\right] \to \frac{1}{2} = 0.5. \quad \text{for } A_0 \sim \mathcal{B}\left(\frac{1}{2}\right)$$

# Average utility

- Scheme 2:




- Average utility:

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u_t\right] \to \frac{5}{8} = 0.625.$$

# Maximal average utility

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} u_t\right] \to \gamma^\star \simeq 0.81$$

where

$$\gamma^\star \quad \text{is the solution of} \quad \frac{h(x)-1}{x-1} = \log_2 3$$

and $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

# Strategies

- Causal case:

$$\begin{array}{rccc} \tau_{i,t}: & \mathcal{S}_i^t \times \mathcal{Y}_i^{t-1} & \to & \mathcal{A}_i \\ & (s_i(1), ..., s_i(t), y_i(1), ..., y_i(t-1)) & \mapsto & a_i(t) \end{array}$$

- Noncausal case:

$$\begin{array}{rccc} \tau_{i,t}: & \mathcal{S}_i^T \times \mathcal{Y}_i^{t-1} & \to & \mathcal{A}_i \\ & (s_i(1), ..., s_i(T), y_i(1), ..., y_i(t-1)) & \mapsto & a_i(t) \end{array}$$

# Important observation

# Important observation

$$
\begin{aligned}
& v_i^{\infty}(\tau_1, \ldots, \tau_K) \\
&= \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[u_i(A_0(t), A_1(t), \ldots, A_K(t))\right] \\
&= \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{a_0, \ldots, a_K} P_{A_0(t), \ldots, A_K(t)}(a_0, \ldots, a_K) u_i(a_0, \ldots a_K) \\
&= \sum_{a_0, \ldots, a_K} u_i(a_0, \ldots a_K) \lim_{T \to +\infty} \frac{1}{T} \sum_{t=1}^{T} P_{A_0(t), \ldots, A_K(t)}(a_0, \ldots, a_K)
\end{aligned}
$$

# Implementable coordination

**Definition** $Q(a_0, a_1, ..., a_K)$ is implementable if $\exists\ (\tau_1, ..., \tau_K)$ s.t.

$$\frac{1}{T}\sum_{t=1}^{T} P_{A_0(t),...,A_K(t)}(a_0, ..., a_K) \to Q(a_0, a_1, ..., a_K)$$

## Characterization of implementable distributions (noncausal case)

**Theorem 1**

– $(A_0(t))_{t \geq 1}$ i.i.d; $A_0 \sim \rho_0$
– $K = 2$
– $A_1(t) = \tau_{1,t}(A_0(1), ..., A_0(T))$
– $A_2(t) = \tau_{2,t}(A_1(1), ..., A_1(t-1))$
► Then $Q(a_0, a_1, a_2)$ is implementable iff its marginal w.r.t $(a_1, a_2)$ is $\rho_0$ and

$$H_Q(A_0, A_1, A_2) \geq H_Q(A_0) + H_Q(A_2).$$

# Performance characterization (Theorem 1)

$$
\begin{array}{lrcl}
\text{minimize} & -\sum_{a_0,a_1,a_2} Q(a_0,a_1,a_2)w(a_0,a_1,a_2) & & \\
\text{subject to} & H_Q(A_0)+H_Q(A_2)-H_Q(A_0,A_1,A_2) & \leq & 0 \\
& -Q(a_0,a_1,a_2) & \leq & 0 \\
& -1+\sum_{a_0,a_1,a_2} Q(a_0,a_1,a_2) & = & 0 \\
& -\rho_0(a_0)+\sum_{a_1,a_2} Q(a_0,a_1,a_2) & = & 0
\end{array}
$$

# Technical challenges

- General case [Larrousse et al ITW 2015]

$$I_Q(S_1; A_2) \leq I_Q(V; Y_2|A_2) + I_Q(V; S_1|A_2)$$

where auxiliary variables are used.

# Characterization of implementable distributions (causal case)

**Theorem 2**

$-$ $(A_0(t))_{t\geq 1}$ i.i.d. + memoryless O.S.
$-$ $K \geq 2$
$-$ $A_i(t) = \tau_{1,t}(S_i(1), ..., S_i(t), Y_i(1), ..., Y_i(t-1))$
► Then $Q(a_0, ..., a_K)$ is implementable iff it factorizes as

$$Q(a_0, ..., a_K) = \sum_{z, s_1, \dots, s_k} \rho_0(a_0) \daleth(s_1, ..., s_k | a_0) P_Z(z) \prod_{k=1}^{K} P_{A_k|S_k,Z}(a_k | s_k, z)$$

[Larrousse et al 2015][Gossner et al 2006]

# Extensions

- Security aspect
- Continuous case
- **Controlled state**

# Stochastic games

**Definition (Stochastic games with individual states):** a stochastic game with individual states is a $6-$uplet $\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_i\}_i, \{\mathcal{X}_i\}_i, \{\widetilde{\mathcal{A}}_i\}_i, \{\alpha_i\}_i, q, \{u_i\}_i\right)$ where

- $\Omega_i$ is the set of individual states of player $i$;
- $\widetilde{\mathcal{A}}_i(x_i)$ is the set of feasible actions for the state $x_i \in \mathcal{X}_i$;
- $\alpha_i : \mathcal{X}_i \to 2^{\mathcal{A}_i}$ is the correspondence determining the feasible actions at a given state of the game;
- under the Markov game assumption, the transition probability of the states is given by:

$$q : \left| \begin{array}{ccc} \mathcal{X} \times \bigotimes_{i=1}^{K} 2^{\mathcal{A}_i} & \to & \Delta(\mathcal{X}) \\ (\underline{x}, \underline{a}) & \mapsto & q(\underline{x}'|\underline{x}, \underline{a}). \end{array} \right.$$

with $\mathcal{X} = \mathcal{X}_1 \times ... \times \mathcal{X}_K$.

# Equilibrium analysis for stochastic games

- Common state with perfect monitoring and recall + finite action and state spaces: there exists an equilibrium in the finitely/discounted repeated games (see Shapley 1953 for $2-$player zero-sum games and Takahashi 1962 Fink 1964 for non-zero-sum games).
- Individual states with perfect monitoring and recall: there exists an equilibrium in the finitely/discounted repeated games (see Vrieze 2007).
- Common state + perfect monitoring + irreducible stochastic games: there is a Folk theorem for infinitely repeated games (Dutta 1991).
- Common state + public signal + irreducible stochastic games: there is a Folk theorem for infinitely repeated games (Hörner etal 2009, Fudenberg and Yamamoto 2009).
- ...

# Differential games (linear-quadratic + common state + finite horizon)

- Control functions: $u_i : t \mapsto u_i(t)$, $i \in \{1, ..., K\}$
- State law:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \mathbf{A}(t)x(t) + \sum_{i=1}^{K} \mathbf{B}_i(t)u_i(t)$$

- Cumulative utility:

$$J_i(u_1, ..., u_K) = \int_{t \in [0,T]} x^{\mathrm{T}}(t)\mathbf{Q}_i x(t)\mathrm{d}t + \sum j = 1^K \int_{t \in [0,T]} u_j^{\mathrm{T}}(t)\mathbf{R}_{ij} u_j(t)\mathrm{d}t + q_i(x_T)$$

# More general differential games

- More general control law:

$$u_i(t, y_i(t))$$

- More general state law:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(t, x(t), u_1(t, y_1(t)), \ldots, u_K(t, y_K(t)))$$

- More general observation structures. Closed-loop perfect state example: $y_i(t) = \{x(t') : 0 \leq t' \leq t\}$. Memoryless perfect state example: $y_i(t) = \{x(0), x(t)\}$.

- Remark (stochastic differential game):

$$\mathrm{d}x(t) = f(t, x(t), u_1(t, y_1(t)), \ldots, u_K(t, y_K(t)))\mathrm{d}t + \mathrm{d}w(t)$$

$\rightarrow$ One path to mean field games.

# Equilibrium analysis for differential/difference games


Basar, T., Olsder, G.J., 1982. Dynamic noncooperative game theory. In: Classics in Applied Mathematics, first ed. SIAM, Philadelphia.

T. Eisele, ”Nonexistence and Nonuniqueness of Open-Loop Equilibria in Linear-Quadratic Differential Games”, Journal of optimization theory and applications, Vol. 37, No. 4, Aug. 1982.

G. Jank, ”Introduction to Non-cooperative Dynamical Game Theory”, Coimbra, March 2001.

G. Jank and H. Abou-Kandil, ”Existence and Uniqueness of Open-Loop Nash Equilibria in Linear-Quadratic Discrete Time Games”, IEEE Trans. on Automatic Control, Vol. 48, No 2, Feb. 2003.

Engwerda, J.C., 2005. LQ Dynamic Optimization and Differential Games. Wiley.

P. Cardaliaguet, ”Introduction to differential games”, Lecture Notes, 2010.

M. Quincampoix, ”Differential games”, Computational complexity. Vols. 16, 854861, Springer, New York, 2012.

# 3. Learning algorithms and strategic-form games

# Algorithm 1: The best-response dynamics (BRD)

**Updating rule (asynchronous BRD)** $K = 2$. Action sequence: $a_1(0), \; a_2(1) \in \mathrm{BR}_2[a_1(0)], \; a_1(2) \in \mathrm{BR}_1[a_2(1)]$, etc. More generally:

$$a_i(t+1) \in \mathrm{BR}_i \left[a_1(t+1), ..., a_{i-1}(t+1), a_{i+1}(t), ..., a_K(t)\right].$$

**Updating rule (synchronous BRD):**

$$a_i(t+1) \in \mathrm{BR}_i \left[a_{-i}(t)\right].$$

[Cournot 1838]

# Comments on Algorithm 1

## Main features

- Fast convergence.

- Steady state: NE.

- Required knowledge: Action profile and individual utility function (in general).

# The iterative water-filling algorithm [Yu et al 2002]

- Actions: $a_i = p_i = (p_{i,1}, ..., p_{i,S})$ with $\sum_s p_{i,s} \leq P^{\max}$ and $p_{i,s} \geq 0$
- BRD:

$$p_i(t+1) \in \text{arg}\max_{p_i} \sum_{s=1}^{S} \log\left(1 + \frac{g_{ii,s}p_{i,s}}{\sigma^2 + \sum_{j \neq i} g_{ji,s}p_{j,s}(t)}\right)$$

- The water-filling solution writes as

$$p_{i,s}(t+1) = \left[\frac{1}{\lambda_i} - \frac{p_i(t)}{\mathrm{SINR}_i(t)}\right]^+$$

# Algorithm 2: Fictitious play (FP)

## Updating rule (synchronous FP):

$$a_i(t+1) \in \text{arg} \max_{a_i \in \mathcal{A}_i} \sum_{a_{-i}} f_{-i,t}(a_{-i}) u_i(a_i, a_{-i}).$$

### Recursive structure

$$\begin{aligned}
f_{i,t+1}(a_i) &= \frac{1}{t+1} \sum_{t'=1}^{t+1} \mathbb{1}_{\{a_{i,t'}=a_i\}} \\
&= \frac{1}{t+1} \sum_{t'=1}^{t} \mathbb{1}_{\{a_{j,t'}=a_j\}} + \frac{1}{t+1} \mathbb{1}_{\{a_{j,t+1}=a_i\}} \\
&= \frac{t}{t+1} f_{i,t}(a_i) + \frac{1}{t+1} \mathbb{1}_{\{a_{j,t+1}=a_i\}} \\
&= f_{i,t}(a_i) + \frac{1}{t+1} \left[ \mathbb{1}_{\{a_{j,t+1}=a_i\}} - f_{i,t}(a_i) \right] \\
&= f_{i,t}(a_i) + \lambda_i(t) \left[ \mathbb{1}_{\{a_{j,t+1}=a_i\}} - f_{i,t}(a_i) \right]
\end{aligned}$$

where $\mathbb{1}$ is the indicator function [Brown 1951].

# Algorithm 3: Reinforcement learning

**A reinforcement learning algorithm.** $|\mathcal{A}_i| < +\infty$,

$\forall i \in \mathcal{K}, \forall n \in \{1, ..., |\mathcal{A}_i|\}$,

$$\pi_i^n(t+1) = \pi_i^n(t) + \lambda_i(t) u_i(t) \left[ 1\!\!1_{\{a_i(t) = a_i^n\}} - \pi_i^n(t) \right],$$

$0 < \lambda_i(t) < 1$.

[Bush and Mosteller 1955][Sastry et al 1994].

## Main features of Algorithm 3

- Required knowledge: individual utility realizations.

- Slow convergence.

- Steady state: NE/boundary points/limit cycle.

# Convergence issue

► Convergence depends on: the updating rule + the associated game.

► For algorithms 1, 2, and 3, it is sufficient that the game be:

□ dominance solvable, or

□ potential, or

□ supermodular.

# An important class of games: Potential games

**Exact potential games [Monderer and Shapley 1996].**
$\exists \Phi, \forall i, \forall s, \forall s'_i,$

$$u_i(s) - u_i(s'_i, s_{-i}) = \Phi(s) - \Phi(s'_i, s_{-i}).$$

**Characterization (special case).** $\mathcal{S}_i = I_i \subset \mathbb{R}$. A game is an exact PG iff:

$$\forall (i,j) \in \mathcal{K}^2, \; \frac{\partial^2 (u_i - u_j)}{\partial s_i \partial s_j} = 0.$$

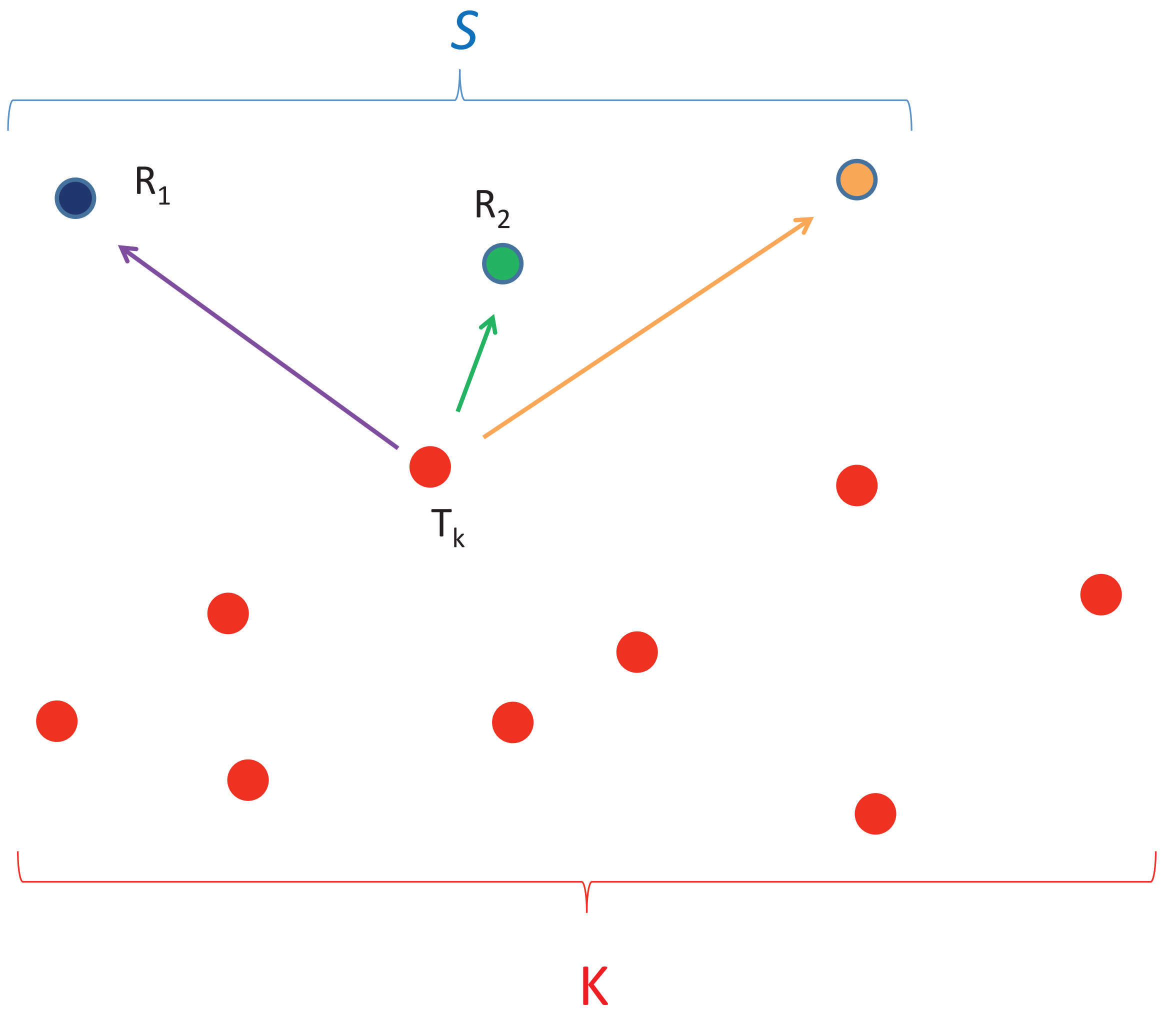
# Potential games. Continued

## Properties

- Convergence of important dynamics. ✓
- Existence of a pure NE. ✓

**Examples.** Team games, dummy games, self-motivated games, congestion games.

# A simple example of potential game



# A simple example of potential game [Perlaza et al 2009]

$$
\begin{aligned}
u_i(p_1, ..., p_K) &= \sum_{s=1}^{S} \log \left( 1 + \frac{g_{i,s} p_{i,s}}{\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s}} \right) \\
&= \sum_{s=1}^{S} \log \left( \frac{\sigma^2 + \sum_{j} g_{j,s} p_{j,s}}{\sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s}} \right) \\
&= \log \left( \sigma^2 + \sum_{j} g_{j,s} p_{j,s} \right) - \log \left( \sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s} \right) \\
&= \Phi(p_1, ..., p_K) - \log \left( \sigma^2 + \sum_{j \neq i} g_{j,s} p_{j,s} \right)
\end{aligned}
$$

# Another important special class of games: Supermodular games

**Definition (supermodularity):** $\mathcal{S}_i$ compact subset of $\mathbb{R}$, $u_i$ upper semi-continuous in $s$, $\forall s_{-i} \geq s'_{-i}$, $u_i(s) - u_i(s_i, s'_{-i})$ is non-decreasing in $s_i$.

**Characterization:**

$$\forall i \neq j, \frac{\partial^2 u_i}{\partial s_i \partial s_j} \geq 0.$$

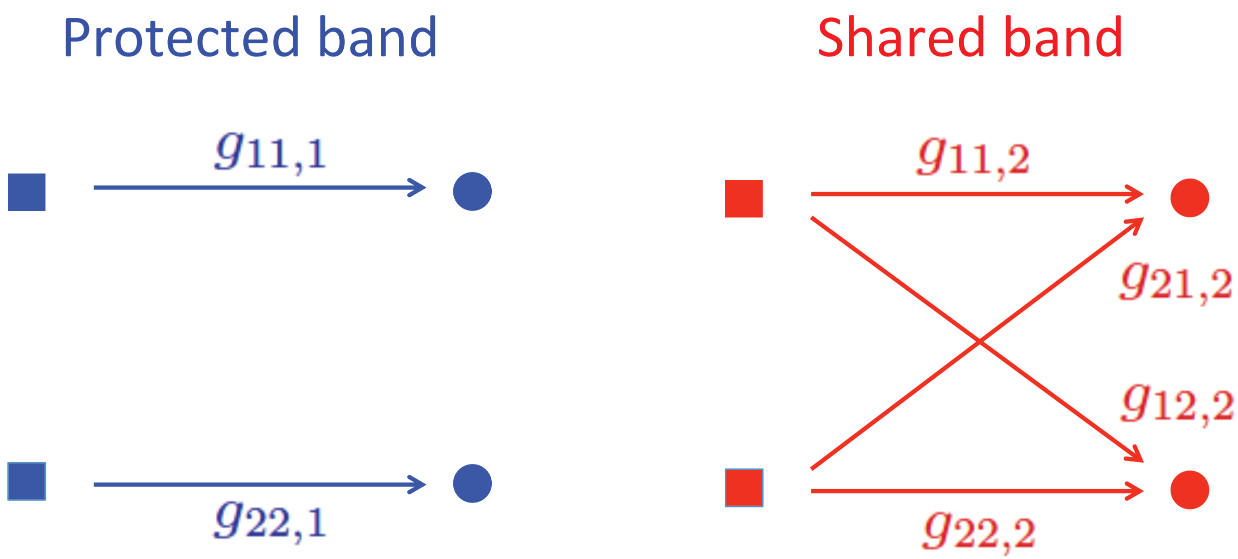
# Supermodular games. Continued

## Properties

- Convergence of important dynamics. $\checkmark$
- Existence of a pure NE. $\checkmark$

**Examples**. Queueing problems [Yao 1995], power control problems [Saraydar et al 2002].

# A simple example of supermodular game [Mochaourab & Jorswieck 2009]




$$u_1(\mu_1, \mu_2) = \log\left(1 + \rho g_{11,1}\mu_1\right) + \log\left(1 + \frac{\rho g_{11,2}\overline{\mu}_1}{1 + \rho g_{21,2}\mu_2}\right)$$

$$u_2(\mu_1, \mu_2) = \log\left(1 + \rho g_{22,1}\overline{\mu}_2\right) + \log\left(1 + \frac{\rho g_{22,2}\mu_2}{1 + \rho g_{12,2}\overline{\mu}_1}\right)$$

## Definition (regret) [Hart & Mas-Collel 2000]

$$\forall n,\ r_{k,a_{k,n}}(t+1) = \frac{1}{t}\sum_{t'=1}^{t} u_k(a_{k,n}, a_{-k}(t')) - u_k(a_k(t'), a_{-k}(t'))$$

## Updating rule

$$\pi_{k,a_{k,n}}(t+1) = \frac{\left[r_{k,a_{k,n}}(t+1)\right]^+}{\sum_{n'=1}^{N_k}\left[r_{k,a_{k,n'}}(t+1)\right]^+}$$

# Main features of Algorithm 4

► Required knowledge: action profile

► Convergence: unconditional convergence + intermediate speed

► Steady state: CCE

Remark: ”pure NE $\subseteq$ mixed NE $\subseteq$ CE $\subseteq$ CCE”

# Coarse correlated equilibrium

## Definition

$$\forall k, \forall a'_k,$$

$$\sum_{a \in \mathcal{A}} q^{\mathrm{CCE}}(a) u_k(a) \geq \sum_{a \in \mathcal{A}} q^{\mathrm{CCE}}(a) u_k(a'_k, a_{-k})$$

## Algorithms to reach a given solution concepts (strategic case)

- **Asynchronous/synchronous best response dynamics, fictitious play, a type of reinforcement algorithm, regret matching,**
- Boltzmann-Gibbs learning,
- coupled dynamics learning,
- trial-and-error learning,
- conditional no-regret learning,
- Bayesian learning,...

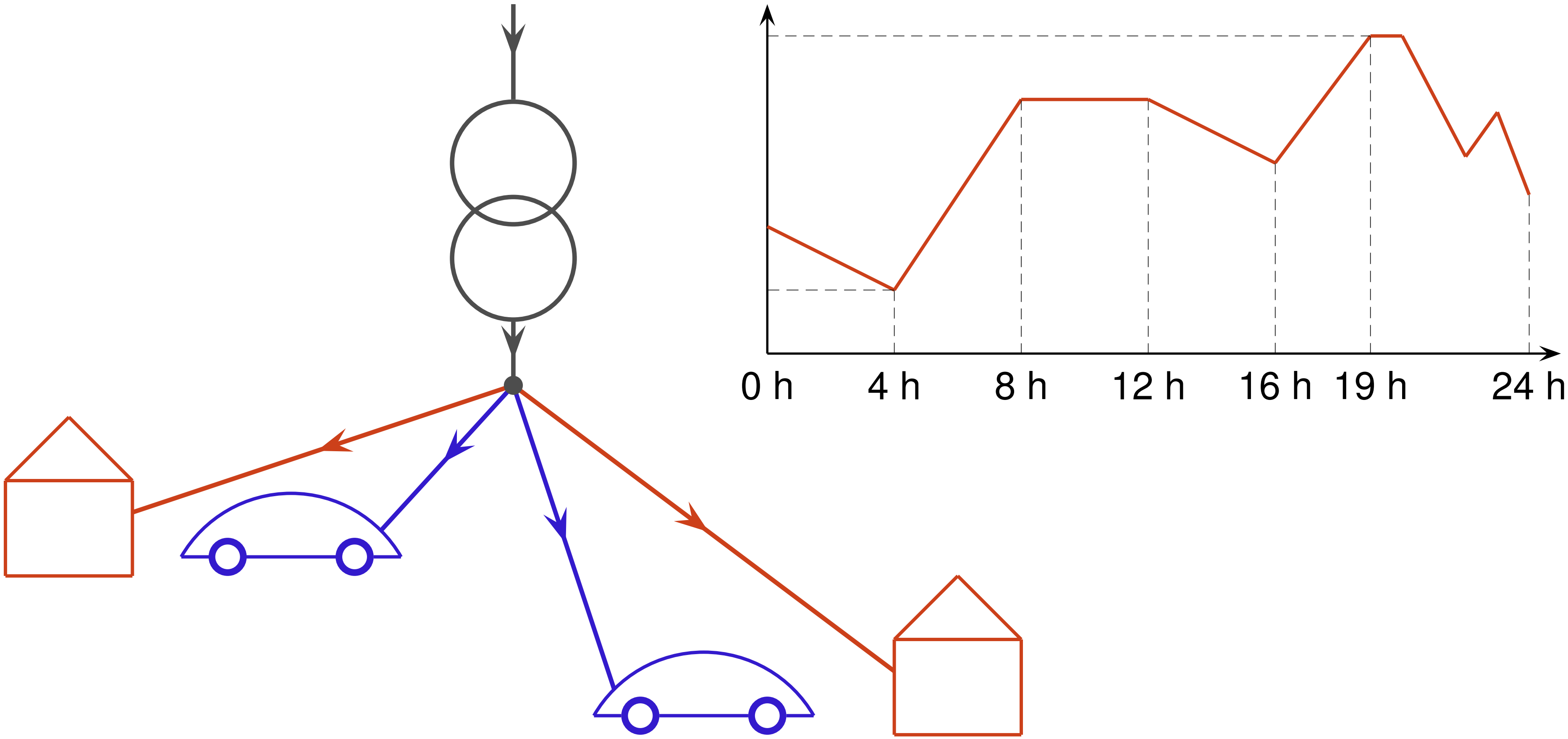
# 4. Case study $\#2$: Power consumption scheduling

## Case study $\#2$: Power consumption scheduling

- Static game formulation.

- A dynamic approach.

# Application example. Continued



# Modeling the problem as a static game

- **Set of players** : $\mathcal{I} = \{1, ..., I\}$.
- **Set of actions** : $s_i \in \mathcal{S}_i = \{1, ..., T\}$
- Action profile : $s = (s_1, ..., s_I)$
- Total load : $\ell_t(s) = \ell_t^{\text{exo}} + \sum_i \ell_{i,t}^{\text{EV}}(s)$
- **Utilities** : $u_i(s) = \sum_{t \in \{s_i, ..., s_i + D_i - 1\}} f_t\left(\ell_1(s), ..., \ell_t(s)\right) + g_i(s_i)$

# Nash equilibrium analysis (1/6)

## Existence

# Nash equilibrium analysis: Existence (2/6)

**Exact potentiality** [Monderer Shapley 1996]

$\exists \Phi, \forall i, \forall s, \forall s_i'$ :

$$u_i(s) - u_i(s_i', s_{-i}) = \Phi(s) - \Phi(s_i', s_{-i})$$

**Ordinal potentiality**

$$u_i(s) - u_i(s_i', s_{-i}) \geq 0 \Leftrightarrow \Phi(s) - \Phi(s_i', s_{-i}) \geq 0$$

Result [Beaude et al TSG 2016]: OP available for memoryless utilities.

# Nash equilibrium analysis (3/6)

**Uniqueness** No

# Nash equilibrium analysis: Determination (4/6)

**Algorithm 1:** The proposed distributed EV charging algorithm.

*Initialize the round index as* $m = 0$. *Initialize the vector of charging start times as* $\mathbf{s}^{(0)}$.

**while** $\|\mathbf{s}^{(m)} - \mathbf{s}^{(m-1)}\| > \delta$ *or* $m \leq M$ **do**

*Outer loop*. *Iterate on the round robin phase index:* $m = m + 1$. *Set* $i = 0$.

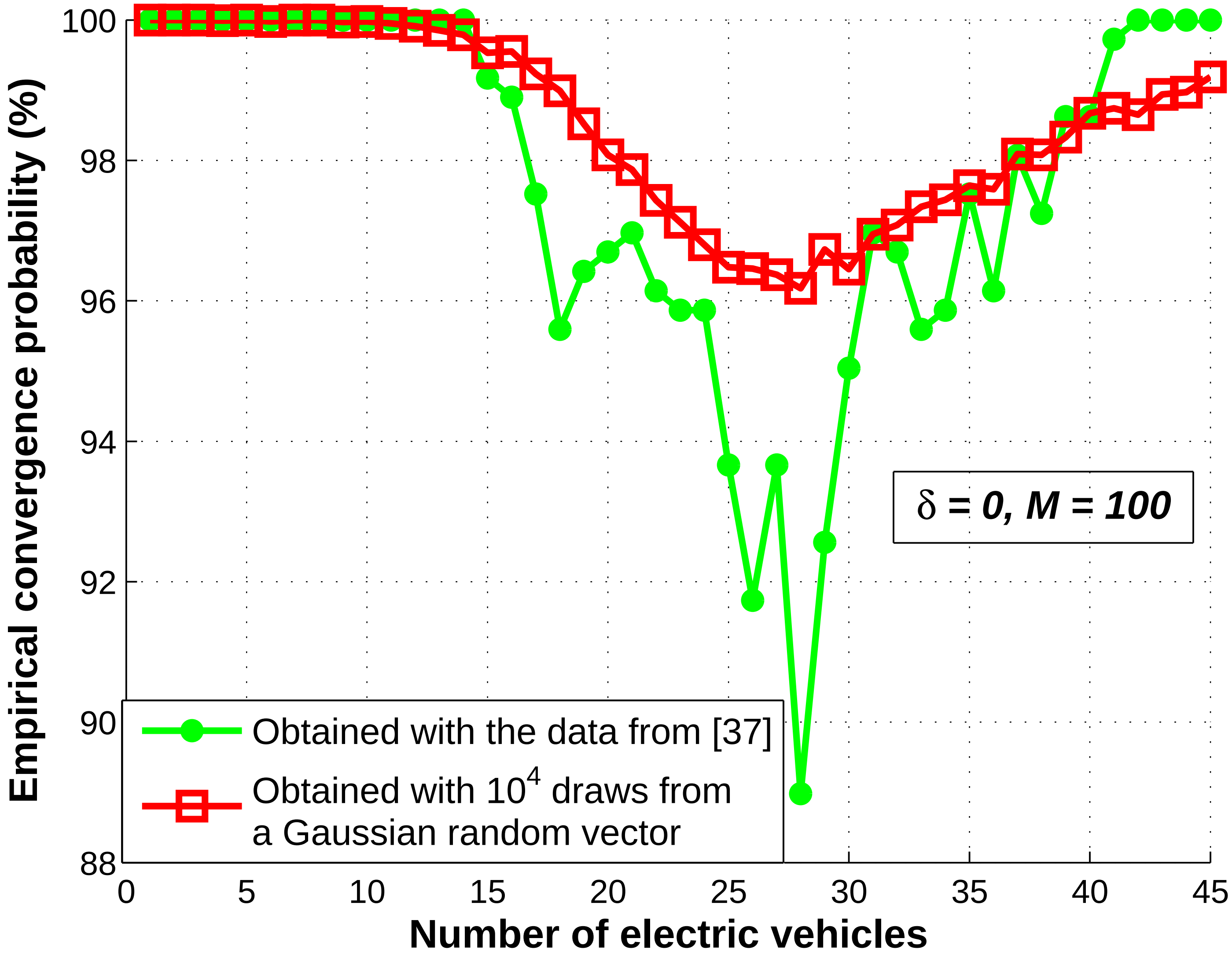
*Inner loop*. *Iterate on the DM index:* $i = i + 1$. *Do:*

$$s_i^{(m)} \in \arg\max_{s_i \in \mathcal{S}_i} u_i(s_1^{(m)}, s_2^{(m)}, \ldots, s_i, s_{i+1}^{(m-1)}, \ldots, s_I^{(m-1)}) \qquad (11)$$

where $s_i^{(m)}(i)$ stands for action of DM $i$ in the round robin phase $m$. Stop when $i = I$ and go to **Outer loop**.

**end**

# Nash equilibrium analysis: BRD convergence (5/6)



# Nash equilibrium analysis: Efficiency (6/6)

**PoA:** PoA $\rightarrow$ 1 when $I \rightarrow \infty$ and under symmetry assumptions. Otherwise, losses may be non-negligible.

$\rightarrow$ Continuous actions

**References:** [Beaude et al Netgcoop 2012][Beaude et al TSG 2016][Paccagnan et al L-CSS 2018]

# Why moving to a dynamical formulation?

# Why moving to a dynamical formulation?


# Objectives for the new formulation: Recap

► Existence of an individual constraint on the state

► More efficiency: discrete actions $\rightarrow$ continuous actions; exploit the dynamical structure

► Directly consider the global cost/utility function

# Optimal control formulation

$$\forall t, \qquad x_t \leq x_{\max}$$

# Optimal control formulation

$$\forall t \qquad x_t = a x_{t-1} + b_1 \times \left( \ell_t^{\text{exo}} + \sum_{i=1}^{I} v_{i,t} \right)^p + b_2 \times \left( \ell_{t-1}^{\text{exo}} + \sum_{i=1}^{I} v_{i,t-1} \right)^q + c_t$$

$$\forall t, \qquad x_t \leq x_{\max} \ ,$$

# Optimal control formulation

$$\forall (i,t), \qquad 0 \leq v_{i,t} \leq V_{\max}$$

$$\forall t \qquad x_t = a x_{t-1} + b_1 \times \left( \ell_t^{\text{exo}} + \sum_{i=1}^{I} v_{i,t} \right)^p + b_2 \times \left( \ell_{t-1}^{\text{exo}} + \sum_{i=1}^{I} v_{i,t-1} \right)^q + c_t$$

$$\forall t, \qquad x_t \leq x_{\max} \,,$$

# Optimal control formulation

$$
\begin{aligned}
\forall i, \qquad & \sum_{t=1}^{T} v_{i,t} \geq C_i \\
\forall (i,t), \qquad & 0 \leq v_{i,t} \leq V_{\max} \\
\forall t \qquad & x_t = a x_{t-1} + b_1 \times \left( \ell_t^{\text{exo}} + \sum_{i=1}^{I} v_{i,t} \right)^p \\
& \qquad + b_2 \times \left( \ell_{t-1}^{\text{exo}} + \sum_{i=1}^{I} v_{i,t-1} \right)^q + c_t \\
\forall t, \qquad & x_t \leq x_{\max} \, ,
\end{aligned}
$$

# Optimal control formulation

minimize $J(v,x) = \sum_{t=1}^{T} e^{\alpha x_t} + \gamma \left( \ell_t^{\text{exo}} + \sum_{i=1}^{I} v_{i,t} \right)$ s.t. :

$$\forall i, \quad \sum_{t=1}^{T} v_{i,t} \geq C_i$$

$$\forall (i,t), \quad 0 \leq v_{i,t} \leq V_{\max}$$

$$\forall t \quad x_t = a x_{t-1} + b_1 \times \left( \ell_t^{\text{exo}} + \sum_{i=1}^{I} v_{i,t} \right)^p + b_2 \times \left( \ell_{t-1}^{\text{exo}} + \sum_{i=1}^{I} v_{i,t-1} \right)^q + c_t$$

$$\forall t, \quad x_t \leq x_{\max} \ ,$$

# Proposed methodology to solve the problem

- Substitution technique for $x_t$
- Operate in a convex regime (e.g., $ab_1 + b_2 \geq 0$)
- Apply the best response dynamics with $v_i = (v_{i,1}, ..., v_{i,T})$

[Beaude et al ECC 2015]

# Stochastic aspects

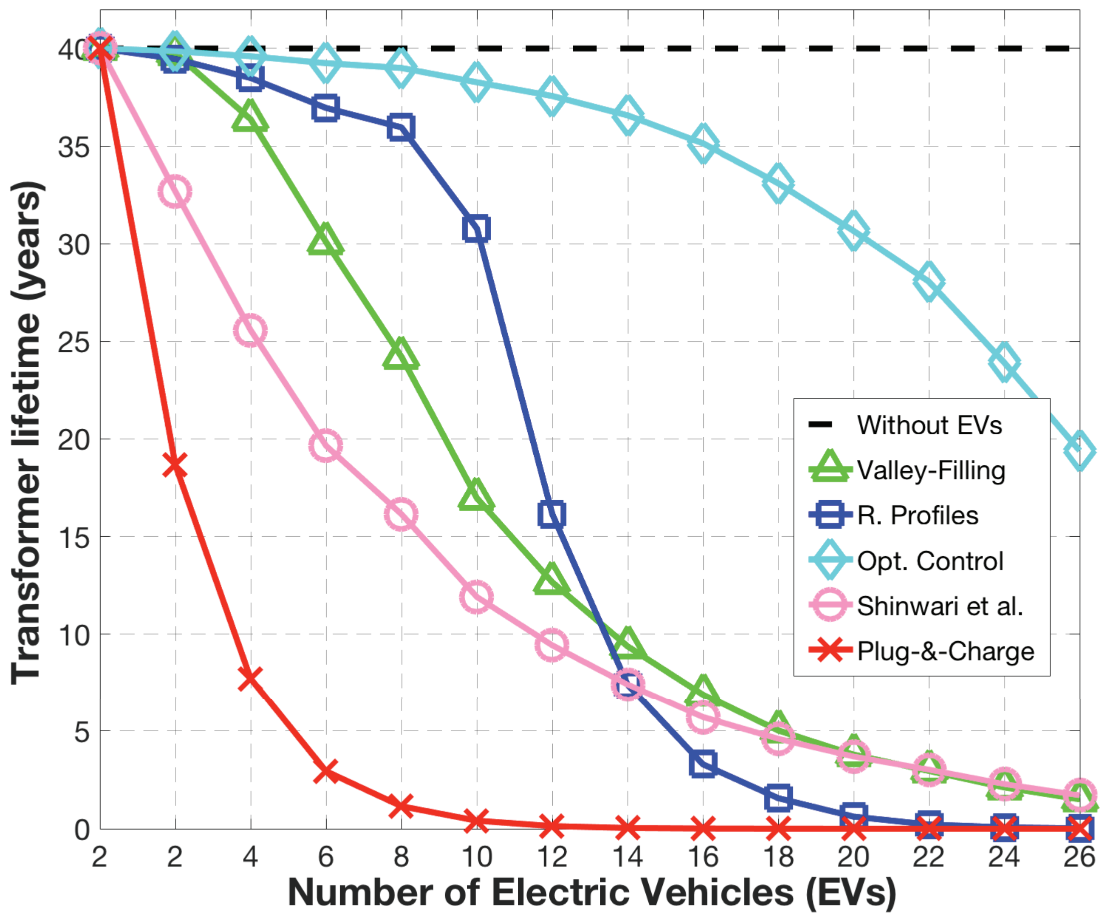
▶ Noisy forecast: $\widetilde{\ell}_t^{\text{exo}} = \ell_t^{\text{exo}} + z_t$

▶ Randomness in the state evolution: $\widetilde{c}_t = c_t + z'_t$

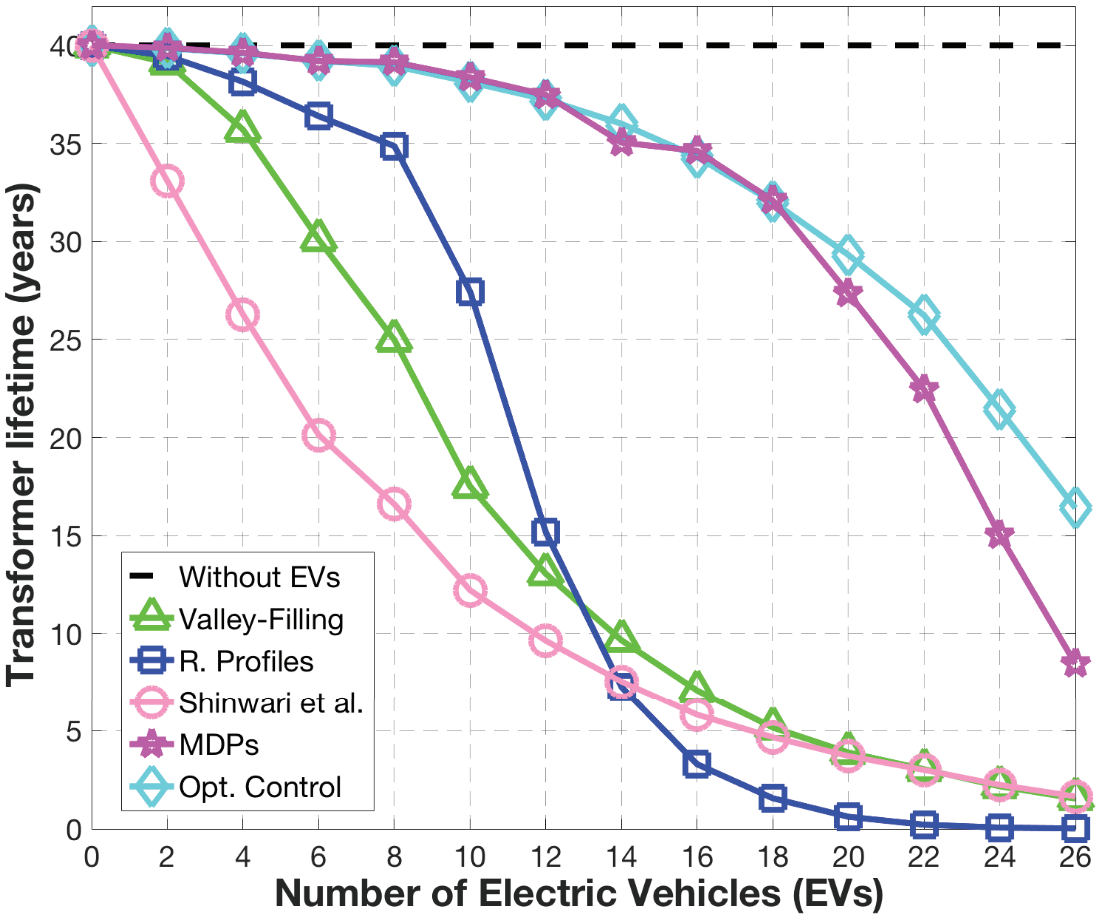
▶ Discretization + apply the best response dynamics with dynamical programming

[Gonzalez et al Gretsi 2017][Gonzalez et al TSG 2018]

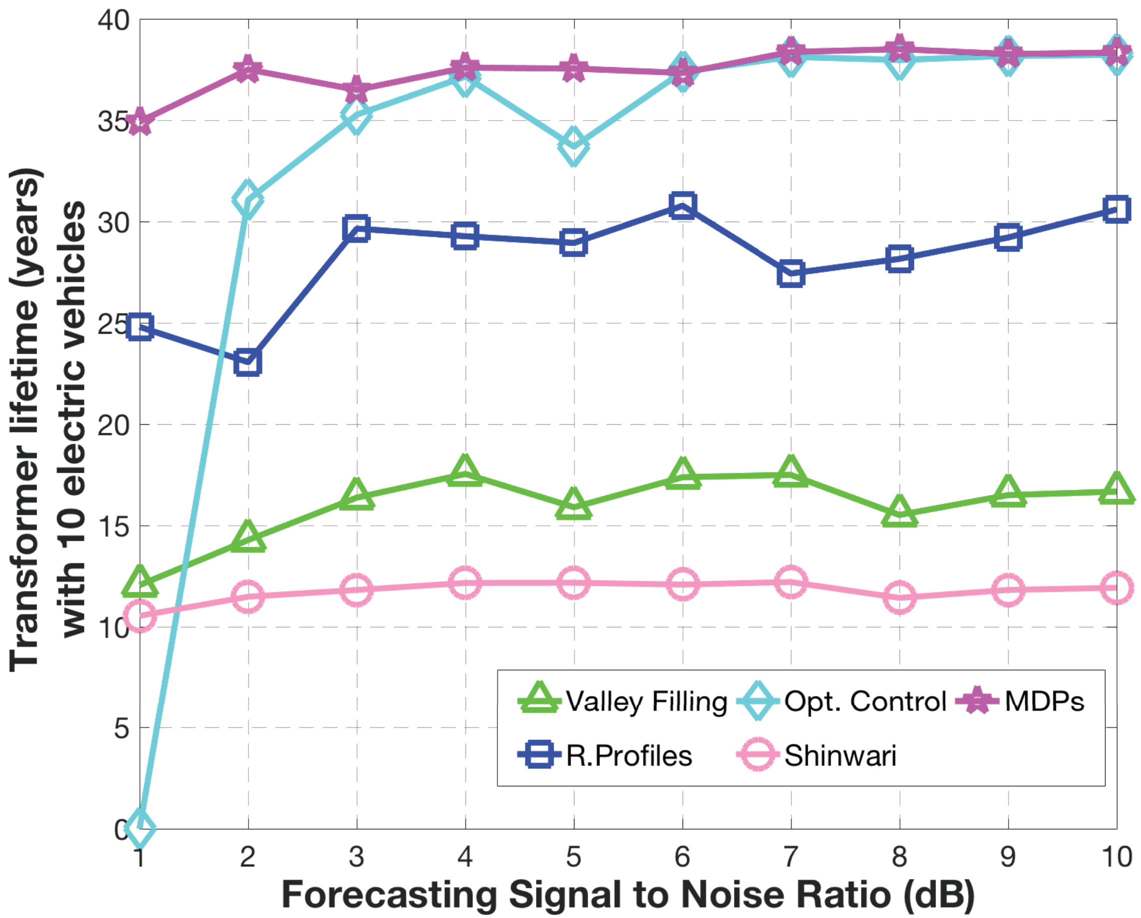
# Illustration 1



# Illustration 2



# Illustration 3



# 4. Coalitional form games

# Moving from strategic-form games to coalition form games

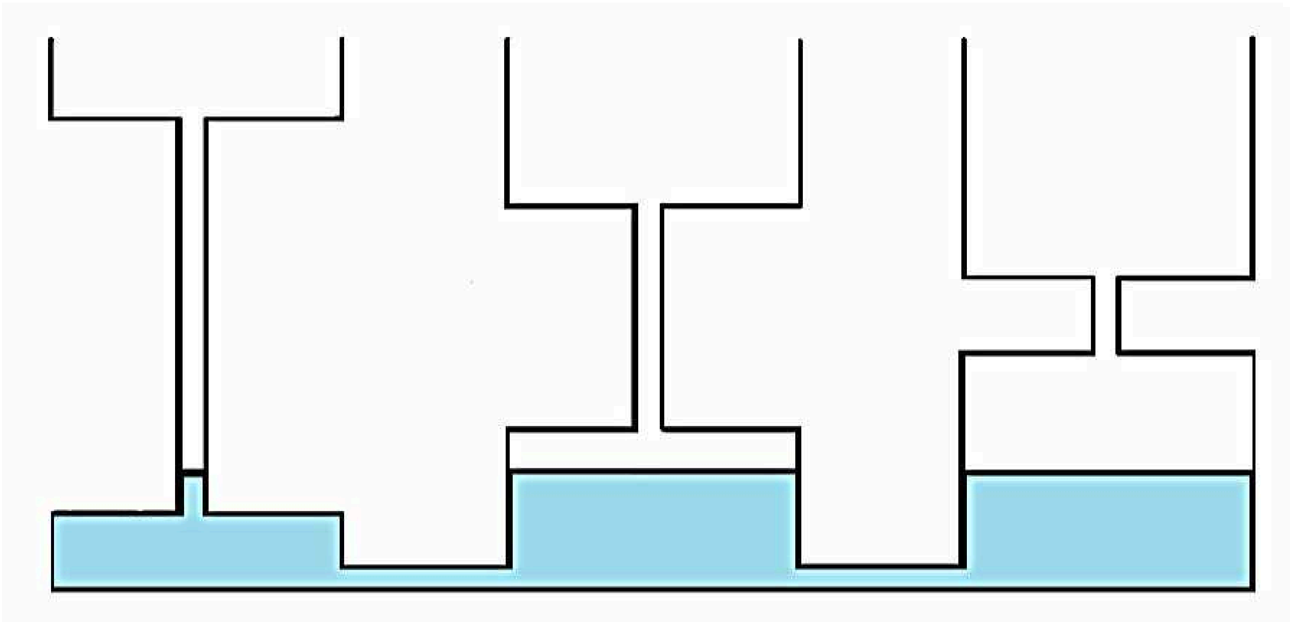
- Cooperation is sought/allowed
- Explicit communication is allowed
- Beyond NBS

## The bankruptcy problem (Talmud's version)

| | | Claim | | |
|---|---|---|---|---|
| | | 100 | 200 | 300 |
| | 100 | $\frac{100}{3}$ | $\frac{100}{3}$ | $\frac{100}{3}$ |
| Estate | 200 | 50 | 75 | 75 |
| | 300 | 50 | 100 | 150 |

## Physical interpretation of the (game-theoretic) solution

[Aumann and Maschler 1985].



**Coalition games can be a very powerful tool.**

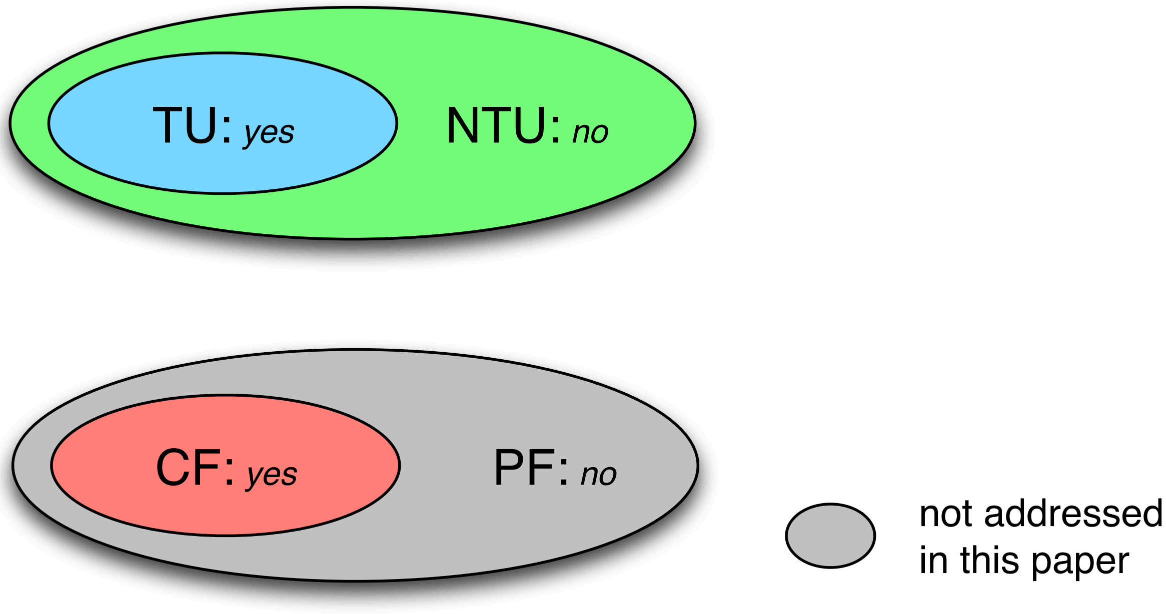
**Two important issues in coalition games:**

- ► **utility allocation/division;**
- ► **coalition formation.**

# Classification of coalition-form games

*Distribution of utility:*
*Can the value of any coalition be divided arbitrarily among its members?*

*Coalition value type:*
*Does the value function of a coalition depend on its own members only?*



# Coalition form games with characteristic functions

**Definition.** Game $\equiv$ pair:

$$\mathcal{G} = (\mathcal{K}, v)\,.$$

**Notation (power set).** Ex:
if $\mathcal{K} = \{1, 2\},\ 2^{\mathcal{K}} = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\}$.

**Transferable utility (TU) games:**

$$\begin{array}{rccc} v: & 2^{\mathcal{K}} & \to & \mathbb{R} \\ & \mathcal{C} & \mapsto & v(\mathcal{C}) \end{array}.$$

# Coalition form games with characteristic functions

## Non transferable utility (NTU) games:

$$\begin{aligned} v \,:\, 2^{\mathcal{K}} &\rightarrow \mathbb{R}^{\mathcal{K}} \\ \mathcal{C} &\mapsto v(\mathcal{C}) = \left\{ \left( v_1(\mathcal{C}), \ldots, v_K(\mathcal{C}) \right) \right\} \end{aligned}.$$

# Ice-cream game example (TU game) $\rightarrow$ investors...

 Chris: \$4,  Marvin: \$3,  Terry: \$3


w = 500

p = \$7


w = 750

p = \$9

- v(Ø) = v({C}) = v({M}) = v({T}) = 0
- v({C, M}) = 500, v({C, T}) = 500, v({M, T}) = 0
- v({C, M, T}) = 750

# Ice-cream game example. General solution concept.

**Utility division:** $x = (x_{\mathrm{C}}, x_{\mathrm{M}}, x_{\mathrm{T}})$.

- $x = (200, 200, 350)$ not stable ($v(\{\mathrm{C}, \mathrm{M}\}) > x_{\mathrm{C}} + x_{\mathrm{M}}$).
- $x' = (250, 250, 250)$ stable.
- $x'' = (750, 0, 0)$ stable.

**Notion of core (TU superadditive games):**

$$\mathrm{core}(\mathcal{G}) = \left\{ x \in \mathbb{R}^K : \sum_{i \in \mathcal{K}} x_i = v(\mathcal{K}),\ \forall \mathcal{C} \subseteq \mathcal{K}, \sum_{i \in \mathcal{C}} x_i \geq v(\mathcal{C}) \right\}.$$

# Ice-cream game core

$$\begin{cases} x_1 + x_2 + x_3 & = & ??? \\ x_1 & \geq & ??? \\ x_2 & \geq & ??? \\ x_3 & \geq & ??? \\ x_1 + x_2 & \geq & ??? \\ x_1 + x_3 & \geq & ??? \\ x_2 + x_3 & \geq & ??? \\ x_1 + x_2 + x_3 & \geq & ??? \end{cases}$$

# Ice-cream game core

$$\begin{cases} x_1 + x_2 + x_3 & = & 750 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \\ x_3 & \geq & 0 \\ x_1 + x_2 & \geq & 500 \\ x_1 + x_3 & \geq & 500 \\ x_2 + x_3 & \geq & 0 \\ x_1 + x_2 + x_3 & \geq & 750 \end{cases}$$

# Core existence: theorems

**Theorem (Bondareva-Shapley)** Not treated here. See e.g., [Bacci et al 2016].

**Definition (convex TU game)**

$$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{K}, \;\; v(\mathcal{C}_1) + v(\mathcal{C}_2) \leq v(\mathcal{C}_1 \cup \mathcal{C}_2) + v(\mathcal{C}_1 \cap \mathcal{C}_2)$$

**Theorem** Convex TU game $\Rightarrow$ non-empty core.

# The nucleolus

## Core

$$\text{core}(\mathcal{G}) = \Big\{x \in \mathbb{R}^K : \sum_{i \in \mathcal{K}} x_i = v(\mathcal{K}),\ \forall \mathcal{C} \subseteq \mathcal{K}, \underbrace{v(\mathcal{C}) - \sum_{i \in \mathcal{C}} x_i}_{e(\mathcal{C},x)} \leq 0\Big\}.$$

**Excess:** $e(x) = (e(\mathcal{C}_1, x), ..., e(\mathcal{C}_{2^K}, x))$ (with $e(\mathcal{C}_1, x) \geq e(\mathcal{C}_2, x) \geq ...$ ).

**Nucleolus (relative to $\mathcal{X} \subseteq \mathbb{R}^K$)**

$$\text{nucleolus}(\mathcal{G}; \mathcal{X}) = \Big\{x \in \mathcal{X} : e(x) \preceq_L e(x'),\ \forall x' \in \mathcal{X}\Big\}.$$

**Motivation** Stability $\rightarrow$ fairness

**Definition** Utility division:

$$x_i = \sum_{\mathcal{C} \subseteq \mathcal{K} \setminus \{i\}} \frac{|\mathcal{C}|!(|\mathcal{K}| - |\mathcal{C}| - 1)!}{|\mathcal{K}|!} \left[ v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right].$$

# Axiomatic characterization

- **Efficiency:** $\sum_{i \in \mathcal{K}} x_i = v(\mathcal{K})$
- **Additivity:** $x_i(\mathcal{G}_1 \oplus \mathcal{G}_2) = x_i(\mathcal{G}_1) + x_i(\mathcal{G}_2$ ($\oplus \equiv v = v_1 + v_2$)
- **Dummy:** $\forall \mathcal{C}', \ v(\mathcal{C}') = v(\mathcal{C}' \cup \{i\})$ ($\mathcal{C}$ does not contain $i$)
- **Symmetry:** $\forall \mathcal{C}'', \ v(\mathcal{C}'' \cup \{i\}) = v(\mathcal{C}'' \cup \{j\})$ ($\mathcal{C}$ does neither contain $i$ nor $j$)

# Coalitional form (NTU)

- **Players:** secondary transmitters $\mathcal{K} = \{1, ..., K\}$.

- **Characteristic function:**

$$v(\mathcal{C}) = 1 - P_{\mathrm{m}}(\mathcal{C}) - J(P_{\mathrm{f}}(\mathcal{C}))$$

with

$$J(P_{\mathrm{f}}(\mathcal{C})) = \left| \begin{array}{ll} -q^2 \log \left[ 1 - \left( \frac{P_{\mathrm{f}}(\mathcal{C})}{q} \right)^2 \right] & \text{if} \quad 0 \leq P_{\mathrm{f}}(\mathcal{C}) < q \\ +\infty & \text{if} \quad q \leq P_{\mathrm{f}}(\mathcal{C}) \leq 1 \end{array} \right. .$$

# Coalition formation

**Utility division.** Not relevant.

**Coalition formation.** Merge and split coalitions by performing Pareto comparisons.

**Results.** Converging algorithm. Distributed solution: implementable, good performance in terms of miss and false alarm probabilities [Saad et al 2011].

## Solution concepts for coalition form games

- **core, nucleolus,**
- $\epsilon-$ core,
- least core,
- kernel,
- bargaining set,
- **Shapley value**, Harsanyi value, Banzhaf index,...

# 5. Extensive form games

# Extensive form games

**Definition:** A standard extensive form game is a $6-$uplet

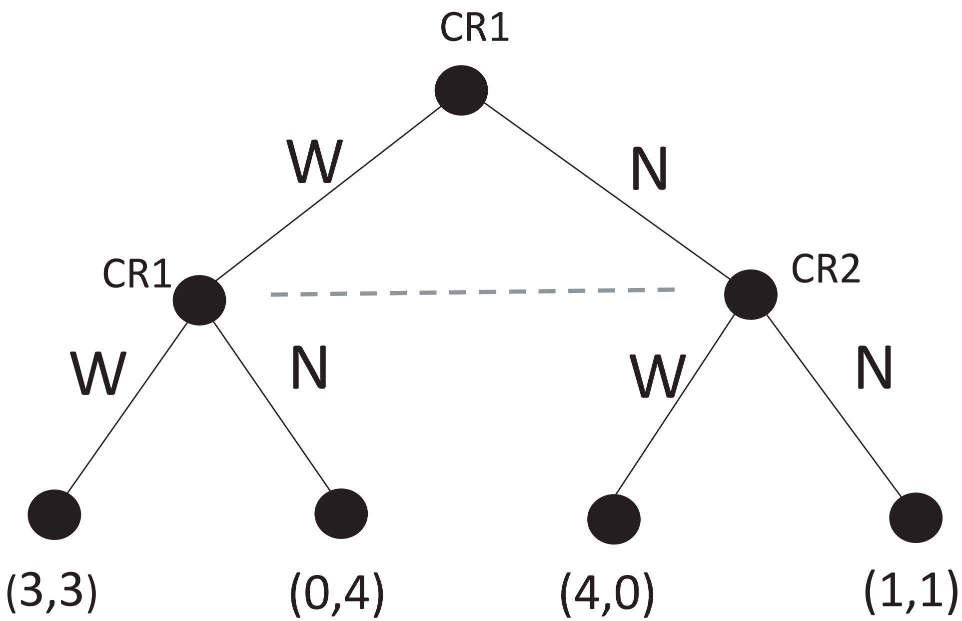
$$\mathcal{G} = (\mathcal{K}, \mathcal{V}, v_{\text{root}}, \pi, \{\mathcal{V}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}})$$

where:

- $\mathcal{K} = \{1, ..., K\}$ is the set of players;
- $(\mathcal{V}, v_{\text{root}}, \pi)$ is a **tree**;
- $\{\mathcal{V}_i\}_{i\in\mathcal{K}}$ is a partition of $\mathcal{V}$.

**Remark:** $\forall v \in \mathcal{V}$, $\exists n \geq 1$, $\pi^{(n)} = \pi \circ ... \circ \pi = v_{\text{root}}$.

# Representing the prisoner's dilemma under extensive form



# Extensive form with imperfect information

**Definition:** It is a $9-$uplet

$$\mathcal{G} = \Big(\mathcal{K}, \mathcal{V}, v_{\mathrm{root}}, \pi, \mathcal{V}_0, \{q_0^j\}_{j\in\mathcal{V}_0}, \{\mathcal{V}_i\}_{i\in\mathcal{K}}, \{W_i^k\}_{k\in\{1,\ldots,k_i\}}, \{u_i\}_{i\in\mathcal{K}}\Big)$$

where:

- player $0$ is nature;
- $\forall j \in \mathcal{V}_0$ $q_j^0$ is the transition probability used by player $0$ to choose a successor to $j$;
- $W_i^k$ corresponds to the partition of $V_i$ which defines the information structure for $i$.

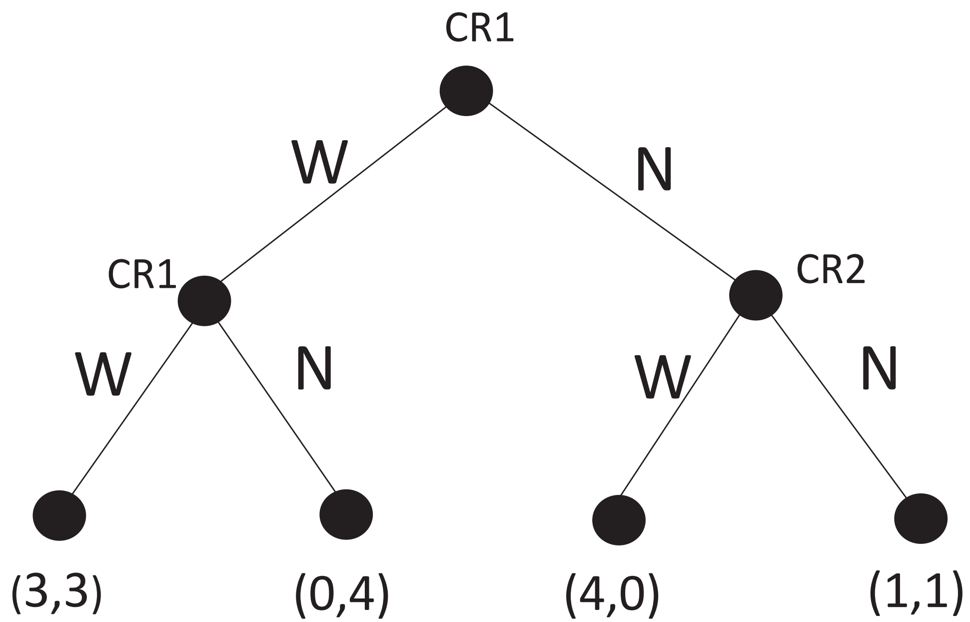
**Remark:** Games with perfect information $W_i^k = \{w_i^k\}$.

# Strategic form and extensive form

- Extensive form **more complete** than strategic form.
- Extensive form usually **less convenient** for mathematical analysis.
- Continuous/discrete action sets.
- Extensive form sometimes more intuitive.
- The tree structure of the extensive form can be useful for computer-based analyses.

# On the difference between static and dynamic games

## Transforming the prisoner's dilemma into a dynamic game



# 6. Conclusion

## Summarizing

► Direct game theory – mechanism design.

► 3 dominant mathematical representations: strategic form, extensive form, coalition form.

► Focus on the Nash equilibrium.

# Summarizing

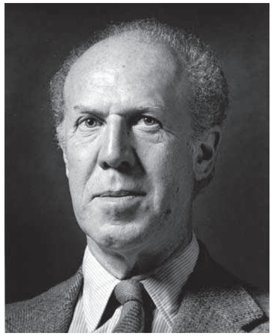
- Static games - dynamic games.
- Relationship between static games and learning.

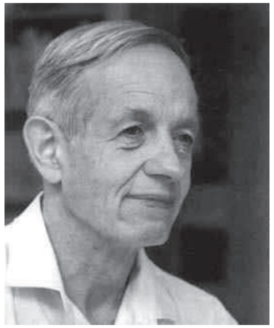
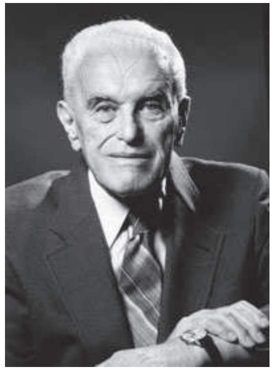
# Challenges

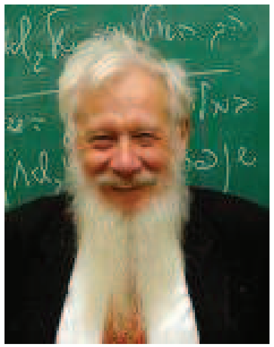
- Tradeoff between efficiency – weak information assumption.
- Bridge the gap between learning and dynamic games.
- Dynamic games with arbitrary observation graphs.
- Mechanism design.

# Mechanism design, Nobel prizes,...





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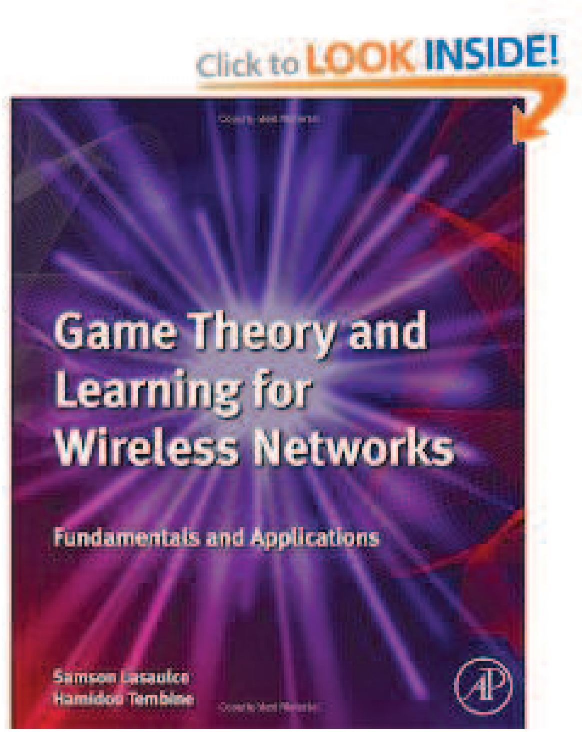

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