Mean Field Game in Principal-Agent Problem

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Mean Field Games

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\[
\sup_a \mathbb{E} \left[ \xi(X^a, \mu^n) - \int_0^T c(t, X^i_t, \mu^n_t, a^i_t) dt \right]
\]

where \( dX^i_t = b(t, X^i_t, \mu^n_t, a^i_t) dt + \sigma(t, X^i_t, \mu^n_t, a^i_t) dW^i_t \)

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What is Mean Field Equilibrium?

A deterministic measure flow \((p_t)_{t \leq T}\) is a MFE if

\[
\begin{align*}
    a^* &= \arg\max_a \mathbb{E}\left[ \xi(X^a, p) - \int_0^T c(t, X^a_t, p_t, a_t) dt \right] \\
    p_t &= \mathcal{L}(X^{a^*}_t)
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\end{cases}
\]

The MFE can be characterized as a fixed point of a PDE system or of a probabilistic model. For example, Carmona and Lacker (15’) identified it as the solution of

\[
\begin{cases}
    dY_t = -H(t, W_t, Z_t, p_t) dt + Z_t dW_t, & Y_T = \xi(W, p) \\
    a^*_t(W) = \nabla_z H(t, W_t, Z_t, p_t) \\
p_t = \mathcal{L}(X^a_t)
\end{cases}
\]

where \(H(t, x, z, p) = \max_{a \in \mathbb{R}} \{az - c(t, x, p, a)\}\).
Principal-Agent Problem with Moral Hazard

Optimal contracting between two parties, when Agent’s effort cannot be observed, is a classical problem in Microeconomics, so-called Principal-Agent problem with moral hazard. It has applications in many areas of economics and finance, for example in corporate governance, portfolio management and energy transition.
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The optimizations of the two parties are coupled through the contract $\xi$:

Agent solves

$$V_0^A = \max_a \mathbb{E} \left[ U \left( \xi(X^a) - \int_0^T c(t, X^a_t, a_t) dt \right) \right]$$

where

$$dX^a_t = a_t dt + dW_t$$

Principal solves

$$V_0^P = \max_{\xi} \mathbb{E} \left[ \tilde{U} \left( X^{a*}, \xi(X^{a*}) \right) \right]$$
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$$V_0^P = \max_{\xi} \mathbb{E} \left[ \tilde{U} \left( X^{a^*}, \xi(X^{a^*}) \right) \right]$$

Mechanism: Principal plays first (giving $\xi$) and then Agent follows (responding by $a^*$). Though Principal CANNOT observe $a$, she CAN predict Agent’s optimal rational response $a^*(\xi)$. 

Zhenjie Ren (CEREMADE)

MFG in PA

Paris, 22/02/2019
Dynamic programming approach to the PA problem

Assume Agent is risk-neutral, i.e. $U = I$. Agent’s value function $V^A$ and his best response $a^*$ can be represented by the solution to the BSDE:

$$dY_t = -H(t, W_t, Z_t)dt + Z_t dW_t, \quad Y_T = \xi(W)$$

$$a^*_t(\xi) = \nabla_z H(t, W_t, Z_t)$$

where $H(t, x, z) = \max_{a \in \mathbb{R}} \{az - c(t, x, a)\}$. 
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where $H(t, x, z) = \max_{a \in \mathbb{R}} \{az - c(t, x, a)\}$. Indeed, if we consider the contracts in the form:

$$\xi \in \Xi := \left\{ Y_T = Y_0 - \int_0^T H(t, W_t, Z_t)dt + Z_t dW_t : \ Y_0 \in \mathbb{R}, Z \in \text{BMO}_2 \right\},$$

the best response of Agent will be $a^*_t = \nabla_z H(t, W_t, Z_t)$. 
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the best response of Agent will be $a^*_t = \nabla_z H(t, W_t, Z_t)$. Therefore, we can use $(Y_0, Z)$ to represent both $\xi$ and $a^*(\xi)$. 
Principal’s optimization

Given the previous representation, we have

\[ V^P_0 = \sup_{\xi \in \Xi} \mathbb{E} \left[ \tilde{U} \left( X^{a*}, \xi(X^{a*}) \right) \right] \]
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= \sup_{Y_0 \in \mathbb{R}, Z \in \text{BMO}_2} \mathbb{E} \left[ \tilde{U} \left( X^{a^*}, Y_T \right) \right], \text{ with } a_t^* = \nabla_z H(t, W_t, Z_t)
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It becomes a classical stochastic control problem where \( Z \) is the control process.
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It becomes a **classical stochastic control problem** where \( Z \) is the control process.

This approach was pioneered by Sannikov (07’) and Cvitanić, Possamaï & Touzi (15’).
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Instead of consider problem involving one Agent, we may consider PA problem with many (infinite) Agents among whom there is mean-field interaction.
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Agent solves

\[ V^A_0 = \max_a \mathbb{E} \left[ U \left( \xi(X^a, p) - \int_0^T c(t, X^a_t, p_t, a_t) \, dt \right) \right] \]

where

\[ dX^a_t = a_t \, dt + dW_t \]

at MFE we have

\[ p_t = \mathcal{L}(X^a_t^*) \]
MFG in PA problem

Instead of consider problem involving one Agent, we may consider PA problem with many (infinite) Agents among whom there is mean-field interaction.

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$$V_0^A = \max_a \mathbb{E}\left[U\left(\xi(X^a, p) - \int_0^T c(t, X^a_t, p_t, a_t)dt\right)\right]$$

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at MFE we have

$$p_t = \mathcal{L}(X^a_t^*)$$

Principal solves

$$V_0^P = \max_\xi \mathbb{E}\left[\tilde{U}\left(X^a_t^*, \xi(X^a_t^*, p), p\right)\right]$$
Consider the contracts in the form:

\[ \Xi := \{ Y_T = Y_0 - \int_0^T H(t, W_t, Z_t, p_t) dt + Z_t dW_t : Y_0 \in \mathbb{R}, Z \in \text{BMO}_2 \}, \]

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where $p_t = \mathcal{L}(X_t)$ for $dX_t = \nabla_z H(t, X_t, Z_t, p_t) dt + dW_t$. It follows from Carmona and Lacker (15’) that for $\xi \in \Xi$, $a_t^* = \nabla_z H(t, W_t, Z_t)$ and $p$ form a MFE.
Same idea... (see Elie, Mastrolia, Possamai 16’)

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$$V_0^P = \sup_{\xi \in \Xi} \mathbb{E}\left[ \tilde{U}\left(X^{a^*}, \xi(X^{a^*}, p), p \right) \right]$$
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$$V_0^P = \sup_{\xi \in \Xi} \mathbb{E} \left[ \tilde{U}(X^{a^*}, \xi(X^{a^*}, p), p) \right]$$

$$= \sup_{Y_0 \in \mathbb{R}, Z \in \text{BMO}_2} \mathbb{E} \left[ \tilde{U}(X^{a^*}, Y_T, p) \right],$$

which is a classical McKean-Vlasov control problem.
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Mean-field Planning Problem

In his course in College de France, P-L. Lions introduced a mean-field planning problem, that is, given two marginal distributions $\mu_0, \mu_1$ on $\mathbb{R}^d$, study the following system:

\[
\begin{align*}
\partial_t u + \frac{1}{2} \Delta u + H(t, x, \nabla u, p) &= 0, \\
\partial_t p - \frac{1}{2} \Delta p + \text{div}(p \nabla_z H(t, x, \nabla u, p)) &= 0, \\
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Let $H(t, x, z, p) = \sup_a \{az - c(t, x, p, a)\}$. By dynamic programming,

$$u(0, X_0) = \sup_a \mathbb{E}\left[u(T, X_T^a) - \int_0^T c(t, X_t^a, p_t, a_t) dt\right], \quad p_t = \mathcal{L}(X_t^a),$$

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where $dX_t^a = a_t dt + dW_t$. Compared to the classical MFG, the terminal condition on $u$ is replaced by the terminal condition on $p$. 
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where $dX_t^a = a_t dt + dW_t$. Compared to the classical MFG, the terminal condition on $u$ is replaced by the terminal condition on $p$.

See e.g. Porretta 14', Orrieri, Porretta, Savaré 18', Graber, Mészáros, Silva, Tonon 18', Benamou, Carlier, Di Marino, Nena 18'.
Let us turn back to the classical MFG:

Given $u_T = \xi$, we are able to find a MFE,
A relaxation related to PA problem

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But WHO is searching for $\xi$? and WHO are playing MFG?
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Here is the corresponding PA problem.

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where \( dX^a_t = a_t dt + dW_t \).

at MFE we have \( p_t = \mathcal{L}(X^{a^*}_t) \).

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In order to solve this PA problem, we relax the contract \( \xi \) to be a path-dependent function.
Advantage of the relaxation

As before, we solve the PA problem by representing \((\xi, a^*(\xi), p)\) by \((Y_0, Z)\)

\[
\xi(W, p) = Y_0 - \int_0^T H(t, W_t, Z_t, p_t) dt + Z_t dW_t \tag{1}
\]

\[
dX_t = \nabla_z H(t, X_t, Z_t, p_t) dt + dW_t \tag{2}
\]

\[
p_t = \mathcal{L}(X_t)
\]
Connection to Mean-Field Planning problem

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Important observation: Given $Z$, (1) and (2) are decoupled!
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$$p_t = \mathcal{L}(X_t)$$

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Note that (2) is a McKean-Vlasov SDE.
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As before, we solve the PA problem by representing \((\xi, a^*(\xi), p)\) by \((Y_0, Z)\)

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\]  

(1)

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\]  

(2)

\[
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\]

**Important observation:** Given \(Z\), (1) and (2) are decoupled! Now the mean-field planning problem becomes searching for \(Z\) s.t. \(\mathcal{L}(X_T) = \mu_1\).

Note that (2) is a McKean-Vlasov SDE. In particular, if \(H(t, x, z, p) = h(t, x, z) + f(t, x, p)\), then

\[
dX_t = \nabla_z h(t, X_t, Z_t) dt + dW_t
\]

is a simpler SDE, which admits weak solution under general conditions.
Solution for linear quadratic model

Consider the LQ model, i.e. $h(t, x, z) = \frac{1}{2} z^2$. Then $dX_t = Z_t dt + dW_t$. 
Solution for linear quadratic model

Consider the LQ model, i.e. \( h(t, x, z) = \frac{1}{2}z^2 \). Then \( dX_t = Z_t dt + dW_t \).

**Proposition (R., Tan, Touzi)**

*If \( \mu_1 \) is equivalent to Leb. measure, then \( \exists Z \) s.t. \( p_T = \mu_1 \).*
Solution for linear quadratic model

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**Proof.** Let \( W^{\mu_0} \) be a B.M. with initial law \( \mu_0 \). Then \( \mathcal{L}(W^{\mu_0}_T) = \mu_0 \ast \mathcal{N}(0, T) \) is equiv. to Leb. measure, and thus \( \mu_1 \) is equiv. to \( \mathcal{L}(W^{\mu_0}_T) \) with a density denoted by \( \varphi \).
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\[
\langle f, \mu_1 \rangle = \langle \varphi f, \mathcal{L}(W^{\mu_0}_T) \rangle = \mathbb{E}[\varphi(W^{\mu_0}_T)f(W^{\mu_0}_T)].
\]
Solution for linear quadratic model

Consider the LQ model, i.e. $h(t, x, z) = \frac{1}{2}z^2$. Then $dX_t = Z_t dt + dW_t$.

Proposition (R., Tan, Touzi)

If $\mu_1$ is equivalent to Leb. measure, then $\exists Z$ s.t. $p_T = \mu_1$.

Proof. Let $W^{\mu_0}$ be a B.M. with initial law $\mu_0$. Then $\mathcal{L}(W_T^{\mu_0}) = \mu_0 \ast \mathcal{N}(0, T)$ is equiv. to Leb. measure, and thus $\mu_1$ is equiv. to $\mathcal{L}(W_T^{\mu_0})$ with a density denoted by $\varphi$. Then

$$\langle f, \mu_1 \rangle = \langle \varphi f, \mathcal{L}(W_T^{\mu_0}) \rangle = \mathbb{E}[\varphi(W_T^{\mu_0})f(W_T^{\mu_0})].$$

Note that $\varphi > 0$ and $\mathbb{E}[\varphi(W_T^{\mu_0})] = 1$, so $\varphi(W_T^{\mu_0})$ can be treated as a change of measure and there exists $Z$ s.t. $\varphi(W_T^{\mu_0}) = \mathcal{D}(Z)$. 
Consider the LQ model, i.e. \( h(t, x, z) = \frac{1}{2} z^2 \). Then \( dX_t = Z_t dt + dW_t \).

**Proposition (R., Tan, Touzi)**

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\[
\langle f, \mu_1 \rangle = \langle \varphi f, \mathcal{L}(W^{\mu_0}_T) \rangle = \mathbb{E}[\varphi(W^{\mu_0}_T)f(W^{\mu_0}_T)].
\]

Note that \( \varphi > 0 \) and \( \mathbb{E}[\varphi(W^{\mu_0}_T)] = 1 \), so \( \varphi(W^{\mu_0}_T) \) can be treated as a change of measure and there exists \( Z \) s.t. \( \varphi(W^{\mu_0}_T) = D(Z) \). Finally it follows from Girsanov theorem that

\[
\mathbb{E}[\varphi(W^{\mu_0}_T)f(W^{\mu_0}_T)] = \mathbb{E}[f(X_T)], \quad \text{for} \quad X_t = W^{\mu_0}_0 + \int_0^t Z_s ds + W_t
\]
Solution for linear quadratic model

Consider the LQ model, i.e. \( h(t, x, z) = \frac{1}{2}z^2 \). Then \( dX_t = Z_t dt + dW_t \).

**Proposition (R., Tan, Touzi)**

*If \( \mu_1 \) is equivalent to Leb. measure, then \( \exists Z \) s.t. \( p_T = \mu_1 \).*

**Proof.** Let \( W_{\mu_0}^T \) be a B.M. with initial law \( \mu_0 \). Then \( \mathcal{L}(W_{\mu_0}^T) = \mu_0 * \mathcal{N}(0, T) \) is equiv. to Leb. measure, and thus \( \mu_1 \) is equiv. to \( \mathcal{L}(W_{\mu_0}^T) \) with a density denoted by \( \varphi \). Then

\[
\langle f, \mu_1 \rangle = \langle f, \mathcal{L}(W_{\mu_0}^T) \rangle = \mathbb{E}[\varphi(W_{\mu_0}^T)f(W_{\mu_0}^T)].
\]

Note that \( \varphi > 0 \) and \( \mathbb{E}[\varphi(W_{\mu_0}^T)] = 1 \), so \( \varphi(W_{\mu_0}^T) \) can be treated as a change of measure and there exists \( Z \) s.t. \( \varphi(W_{\mu_0}^T) = \mathcal{D}(Z) \). Finally it follows from Girsanov theorem that

\[
\mathbb{E}[\varphi(W_{\mu_0}^T)f(W_{\mu_0}^T)] = \mathbb{E}[f(X_T)], \quad \text{for} \quad X_t = W_{\mu_0}^0 + \int_0^t Z_s ds + W_t
\]
Some extensions

Here are some feasible extensions:

- Let the volatility be controlled: \( dX_t = a_t dt + \sigma_t dW_t \)

- Since \( \{\xi : p_T = \mu_1\} \neq \emptyset \), Principal can further study the optimal transport problem:

\[
V_0^P = \max_{\xi : p_T = \mu_1} \mathbb{E}[\tilde{U}(X^{a^*}, \xi, p)].
\]

This is a generalization to the (semi-)martingale transport problem.
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Many job opportunities
Agent can fire Boss...

Consider a model with **TWO Principals** but one Agent. Allow Agent to switch working for different Principals.
Agent can fire Boss...

Consider a model with **TWO Principals** but one Agent. Allow Agent to switch working for different Principals.

- Let $I_t \in \{0,1\}$ (right cont.) record for whom Agent works at time $t$.  

\[
\text{Controlled output } \quad dX_t = \left(d(X_0^t, X_1^t)\right)^T = b(t, X_t, \lambda_{I_t}) dt + dW_t \text{, e.g.}
\]

Let $I$ be a Poisson point proc. s.t. the first jump time $\tau$ follows the conditional law $P[\tau \geq t | F_{X_t, I_t}] = e^{-\int_0^t \lambda_s ds}$.  

The intensity proc. $\lambda$ describes the hesitation of changing employer. The bigger $\lambda$ is, the less hesitation Agent has. We shall allow Agent to control $I$ through choosing $\lambda$.  

Agent can fire Boss...

Consider a model with **TWO Principals** but one Agent. Allow Agent to switch working for different Principals.

- Let $I_t \in \{0, 1\}$ (right cont.) record for whom Agent works at time $t$.
- Controlled output $dX_t = d(X_t^0, X_t^1)^\top = b(t, X_t, a_t^I, I_t)dt + dW_t$, e.g.

  $$b(t, x, a^0, 0) = \begin{pmatrix} a^0 \\ 0 \end{pmatrix} \quad \text{and} \quad b(t, x, a^1, 1) = \begin{pmatrix} 0 \\ a^1 \end{pmatrix}$$
Agent can fire Boss...

Consider a model with **TWO Principals** but one Agent. Allow Agent to switch working for different Principals.

- Let $l_t \in \{0, 1\}$ (right cont.) record for whom Agent works at time $t$.
- Controlled output $dX_t = d(X^0_t, X^1_t)^\top = b(t, X_t, a^l_t, l_t) dt + dW_t$, e.g.

$$b(t, x, a^0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad b(t, x, a^1, 1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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\[
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\]

- Let $l$ be a Poisson point proc. s.t. the first jump time $\tau$ follows the conditional law $\mathbb{P}[\tau \geq t | \mathcal{F}_t, l] = e^{-\int_0^t \lambda_s ds}$. 
Consider a model with **TWO Principals** but one Agent. Allow Agent to switch working for different Principals.

- Let $I_t \in \{0, 1\}$ (right cont.) record for whom Agent works at time $t$.
- Controlled output $dX_t = d(X^0_t, X^1_t) = b(t, X_t, a^I_t, I_t)dt + dW_t$, e.g.

$$b(t, x, a^0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad b(t, x, a^1, 1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Let $I$ be a Poisson point proc. s.t. the first jump time $\tau$ follows the conditional law $\mathbb{P}[\tau \geq t|\mathcal{F}_t, I] = e^{-\int_0^t \lambda s ds}$. The intensity proc. $\lambda$ describes the hesitation of changing employer. The bigger $\lambda$ is, the less hesitation Agent has. We shall allow Agent to control $I$ through choosing $\lambda$. 
Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

$$V^A_0 = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{\{I_T=0\}} + \xi^1 1_{\{I_T=1\}} - \int_0^T (c(t, X_t, a^I, l_t) + \frac{1}{2} |\lambda^I_t|^2) dt]$$
Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

$$V^A_0 = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{I_T=0} + \xi^1 1_{I_T=1} - \int_0^T (c(t, X_t, a^t_t, l_t) + \frac{1}{2} |\lambda^t_t|^2) dt]$$

Denote by $V^A_{t,0} := V^A_t |_{l_t=0}$ and $V^A_{t,1} := V^A_t |_{l_t=1}$. 
Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

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Denote by $V^A_{t,0} := V^A_t |_{I_t=0}$ and $V^A_{t,1} := V^A_t |_{I_t=1}$. By DPP:

$$\begin{cases} V^A_{0,0} = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{\{\tau>T\}} + V^A_{\tau,1} 1_{\{\tau\leq T\}} - \int_0^{T\wedge \tau} \frac{1}{2} |\lambda^0_t| dt] \\ V^A_{0,1} = \max_{a, \lambda} \mathbb{E}[\xi^1 1_{\{\tau>T\}} + V^A_{\tau,0} 1_{\{\tau\leq T\}} - \int_0^{T\wedge \tau} \frac{1}{2} |\lambda^1_t| dt] \end{cases}$$

Note that $P[\tau \geq t | F_{X, I_t}] = e^{-\int_0^t \lambda_s ds} =: \Lambda_t$. This becomes classical control problem characterized by the BSDE system:

$$\begin{align*} dY^0_t &= -dt + Z^0_t dW_t, \quad Y^0_T = \xi^0 \\ dY^1_t &= -dt + Z^1_t dW_t, \quad Y^1_T = \xi^1 \end{align*}$$
Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

$$V^A_0 = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{\{I_T=0\}} + \xi^1 1_{\{I_T=1\}} - \int_0^T \left( c(t, X_t, a^l_t, I_t) + \frac{1}{2} |\lambda^l_t|^2 \right) dt]$$

Denote by $V^A_{t,0} := V^A_t|_{I_t=0}$ and $V^A_{t,1} := V^A_t|_{I_t=1}$. By DPP:

$$\begin{cases} V^A_{0,0} = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{\{\tau>T\}} + V^A_{\tau,1} 1_{\{\tau\leq T\}} - \int_0^{T^\wedge \tau} \frac{1}{2} |\lambda^0_t|^2 dt] \\
V^A_{0,1} = \max_{a, \lambda} \mathbb{E}[\xi^1 1_{\{\tau>T\}} + V^A_{\tau,0} 1_{\{\tau\leq T\}} - \int_0^{T^\wedge \tau} \frac{1}{2} |\lambda^1_t|^2 dt] \end{cases}$$

Note that $\mathbb{P}[\tau \geq t \mid \mathcal{F}^X_t] = e^{-\int_0^t \lambda_s ds} =: \Lambda_t$. 

Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

\[
V_0^A = \max_{a, \lambda} \mathbb{E}\left[ \xi^0 1_{\{I_T=0\}} + \xi^1 1_{\{I_T=1\}} - \int_0^T \left( c(t, X_t, a^t_l, l_t) + \frac{1}{2} |\lambda^t_l|^2 \right) dt \right]
\]

Denote by $V_t^{A,0} := V_t^A|_{l_t=0}$ and $V_t^{A,1} := V_t^A|_{l_t=1}$. By DPP:

\[
\begin{align*}
V_0^{A,0} &= \max_{a, \lambda} \mathbb{E}\left[ \Lambda_T \xi^0 + \int_0^T \Lambda_t (\lambda^0_t V_t^{A,1} - \frac{1}{2} |\lambda^0_t|^2) dt \right] \\
V_0^{A,1} &= \max_{a, \lambda} \mathbb{E}\left[ \Lambda_T \xi^1 + \int_0^T \Lambda_t (\lambda^1_t V_t^{A,1} - \frac{1}{2} |\lambda^1_t|^2) dt \right]
\end{align*}
\]

Note that $\mathbb{P}[\tau \geq t | \mathcal{F}_t^{X,l}] = e^{-\int_0^t \lambda_s ds} =: \Lambda_t$. 
Agent’s problem: optimal switching

Given two contracts $\xi^0, \xi^1$, Agent solves the optimal switching problem

$$V_A^0 = \max_{a, \lambda} \mathbb{E} [\xi^0 1\{I_T=0\} + \xi^1 1\{I_T=1\} - \int_0^T (c(t, X_t, a^0_t, I_t) + \frac{1}{2}|\lambda^0_t|^2) \, dt]$$

Denote by $V_{t,0}^A := V_t^A|_{I_t=0}$ and $V_{t,1}^A := V_t^A|_{I_t=1}$. By DPP:

$$\begin{align*}
V_{0,0}^A &= \max_{a, \lambda} \mathbb{E} \left[ \Lambda_T \xi^0 + \int_0^T \Lambda_t (\lambda^0_t V_{t,1}^A - \frac{1}{2}|\lambda^0_t|^2) \, dt \right] \\
V_{0,1}^A &= \max_{a, \lambda} \mathbb{E} \left[ \Lambda_T \xi^1 + \int_0^T \Lambda_t (\lambda^1_t V_{t,1}^A - \frac{1}{2}|\lambda^1_t|^2) \, dt \right]
\end{align*}$$

Note that $\mathbb{P}[\tau \geq t|\mathcal{F}_t^{X,I}] = e^{-\int_0^t \lambda_s \, ds} =: \Lambda_t$. This becomes classical control problem characterized by the BSDE system

$$\begin{align*}
dY_t^0 &= -\max_{\lambda \geq 0} \{ -\lambda Y_t^0 + \lambda Y_t^1 - \frac{1}{2}\lambda^2 + Z_t^0 \} \, dt + Z_t^0 \, dW_t, \quad Y_T^0 = \xi^0 \\
dY_t^1 &= -\max_{\lambda \geq 0} \{ -\lambda Y_t^1 + \lambda Y_t^0 - \frac{1}{2}\lambda^2 + Z_t^1 \} \, dt + Z_t^1 \, dW_t, \quad Y_T^1 = \xi^1
\end{align*}$$
Agent’s problem: optimal switching

Given two contracts \(\xi^0, \xi^1\), Agent solves the optimal switching problem

\[
V^A_0 = \max_{a, \lambda} \mathbb{E}[\xi^0 1_{\{I_T=0\}} + \xi^1 1_{\{I_T=1\}} - \int_0^T (c(t, X_t, a^l_t, l_t) + \frac{1}{2}|\lambda^l_t|^2) dt]
\]

Denote by \(V^A_{t,0} := V^A_t|_{I_t=0}\) and \(V^A_{t,1} := V^A_t|_{I_t=1}\). By DPP:

\[
\begin{cases}
V^A_{0,0} = \max_{a, \lambda} \mathbb{E}[\Lambda_T \xi^0 + \int_0^T \Lambda_t (\lambda^0_t V^A_{t,1} - \frac{1}{2}|\lambda^0_t|^2) dt]

V^A_{0,1} = \max_{a, \lambda} \mathbb{E}[\Lambda_T \xi^1 + \int_0^T \Lambda_t (\lambda^1_t V^A_{t,1} - \frac{1}{2}|\lambda^1_t|^2) dt]
\end{cases}
\]

Note that \(\mathbb{P}[\tau \geq t|\mathcal{F}^X_t] = e^{-\int_0^t \lambda_s ds} =: \Lambda_t\). This becomes classical control problem characterized by the BSDE system

\[
\begin{cases}
dY^0_t = -\left(\frac{1}{2}(Y^1_t - Y^0_t)^2 + Z^0_t\right) dt + Z^0_t dW_t, \quad Y^0_T = \xi^0

dY^1_t = -\left(\frac{1}{2}(Y^0_t - Y^1_t)^2 + Z^1_t\right) dt + Z^1_t dW_t, \quad Y^1_T = \xi^1
\end{cases}
\]
Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts $(\xi^0, \xi^1)$ by $(Y^0_0, Y^1_0, Z^0, Z^1)$, i.e. consider the contracts in

$$
\Xi = \{ \xi^0 = Y^0_0 + \int_0^T \left( \frac{1}{2} (Y^1_t - Y^0_t)^2 + Z^0_t \right) dt + Z^0_t dW_t \\
\xi^1 = Y^1_0 + \int_0^T \left( \frac{1}{2} (Y^0_t - Y^1_t)^2 + Z^1_t \right) dt + Z^1_t dW_t \}.
$$
Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts \((\xi^0, \xi^1)\) by \((Y^0_0, Y^1_0, Z^0, Z^1)\), i.e. consider the contracts in

\[
\Xi = \left\{ \begin{array}{l}
\xi^0 = Y^0_0 + \int_0^T \left( \frac{1}{2}(Y^1_t - Y^0_t)^2 + Z^0_t \right) dt + Z^0_t dW_t \\
\xi^1 = Y^1_0 + \int_0^T \left( \frac{1}{2}(Y^0_t - Y^1_t)^2 + Z^1_t \right) dt + Z^1_t dW_t
\end{array} \right\}
\]

Under these contracts, Agent’s optimal intensity is

\[
\lambda^* = (Y^1_t - Y^0_t)1_{\{l_t=0\}} + (Y^0_t - Y^1_t)1_{\{l_t=1\}}
\]
Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts \((\xi^0, \xi^1)\) by \((Y_0^0, Y_0^1, Z^0, Z^1)\), i.e. consider the contracts in

\[
\Xi = \left\{ \xi^0 = Y_0^0 + \int_0^T \left( \frac{1}{2}(Y_t^1 - Y_t^0)^2 + Z_t^0 \right) dt + Z_t^0 dW_t \right. \\
\xi^1 = Y_0^1 + \int_0^T \left( \frac{1}{2}(Y_t^0 - Y_t^1)^2 + Z_t^1 \right) dt + Z_t^1 dW_t \right\}.
\]

Under these contracts, Agent’s optimal intensity is

\[\lambda^*_t = (Y_t^1 - Y_t^0) + 1_{\{I_t=0\}} + (Y_t^0 - Y_t^1) + 1_{\{I_t=1\}}\]

So Principal’s problem becomes

\[V_{0,i}^P = \max_{\xi^i \in \Xi} \mathbb{E}[U(X_T^i - \xi^i 1_{\{I_T=i\}})], \quad i \in \{0, 1\}\]
Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts \((\xi^0, \xi^1)\) by \((Y^0_0, Y^1_0, Z^0, Z^1)\), i.e. consider the contracts in

$$\begin{align*}
\Xi &= \left\{ \xi^0 = Y^0_0 + \int_0^T \left( \frac{1}{2}(Y^1_t - Y^0_t)^2 + Z^0_t \right) dt + Z^0_t dW_t \\
\xi^1 &= Y^1_0 + \int_0^T \left( \frac{1}{2}(Y^0_t - Y^1_t)^2 + Z^1_t \right) dt + Z^1_t dW_t \right\}.
\end{align*}$$

Under these contracts, Agent’s optimal intensity is

$$\lambda^*_t = (Y^1_t - Y^0_t) + 1\{I_t = 0\} + (Y^0_t - Y^1_t) + 1\{I_t = 1\}$$

So Principal’s problem becomes

$$V^P_{0,i} = \max_{Y^0_i, Z^i} \mathbb{E}[U(X^i_T - \xi^i 1\{I_T = i\})], \quad i \in \{0, 1\}$$
Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts \((\xi^0, \xi^1)\) by \((Y^0_0, Y^1_0, Z^0, Z^1)\), i.e. consider the contracts in

\[
\begin{align*}
\Xi &= \left\{ \begin{array}{l}
\xi^0 = Y^0_0 + \int_0^T \left( \frac{1}{2}(Y^1_t - Y^0_t)^2 + Z^0_t \right) dt + Z^0_t dW_t \\
\xi^1 = Y^1_0 + \int_0^T \left( \frac{1}{2}(Y^0_t - Y^1_t)^2 + Z^1_t \right) dt + Z^1_t dW_t 
\end{array} \right. \\
\end{align*}
\]

Under these contracts, Agent’s optimal intensity is

\[
\lambda^*_t = (Y^1_t - Y^0_t) + 1_{\{I_t = 0\}} + (Y^0_t - Y^1_t) + 1_{\{I_t = 1\}}
\]

So Principal’s problem becomes

\[
V_{0, i}^{P} = \max_{Y^i_0, Z^i} E[U(X^i_T - \xi^i 1_{\{I_T = i\}})], \quad i \in \{0, 1\}
\]

However, since \(Z\) does NOT dependent on \(I\), this optimization is time-inconsistent.
More principals...

Consider the same model but with $n$ Principals. After an appropriate normalization, the Agent’s problem can be characterized by the system of BSDE:

$$dY_t^i = -\left(\frac{1}{2(n-1)} \sum_{j \neq i} (Y_t^j - Y_t^i)^2 + Z_t^i\right)dt + Z_t^i dW_t, \quad Y_T^0 = \xi^i, \quad 1 \leq i \leq n,$$

and the optimal intensity of switching to $j$-th Principal is $\lambda_t^{j,*} = \frac{(Y_t^j - Y_t^i)_+}{n-1}$.

Let $n \rightarrow \infty$. Heuristically, the equation converges to

$$dY_t = -\left(\frac{1}{2} \int (y - Y_t)_+ p_t(dy) + Z_t\right)dt + Z_t dW_t, \quad p_t = \mathcal{L}(Y_t)$$

and once Agent leaves a company, there is NO chance he comes back.
MFG among Principals

Given \( \{p_t\} \), consider the contract represented by \( Y_0, Z \):

\[
\xi \in \Xi(p) = \left\{ Y_0 - \int_0^T \left( \frac{1}{2} \int (y - Y_t)^2 p_t(dy) + Z_t \right) dt + \int_0^T Z_t dW_t \right\}
\]

Each Principal faces the optimization:

\[
\lambda_t = \int (y - Y_t) + p_t(dy) \text{ and } \Lambda_t = e^{-\int_0^t \lambda_s ds}.
\]

Once \( L(Y_\ast_t) = p_t \), the Principals reach a MFE.

Proposition (Hu, R., Yang) Further assume \( |Z|_\infty \leq C \). Then there exists such MFE.
MFG among Principals

Given \( \{p_t\} \), consider the contract represented by \( Y_0, Z \):

\[
\xi \in \Xi(p) = \left\{ Y_0 - \int_0^T \left( \frac{1}{2} \int (y - Y_t)^2 p_t(dy) + Z_t \right) dt + \int_0^T Z_t dW_t \right\}
\]

Each Principal faces the optimization:

\[
V_{0}^{P,i}(p) = \max_{Y_0^i, Z_i^i} \mathbb{E}[X_T^i - \xi^i 1_{\{I_T=i\}}]
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\]

\[
= \begin{cases} 
\mathbb{E}[X_{0}^{i} + W_{T}] = X_{0}^{i}, & \text{as } l_0 \neq i \\
\max_{Y_{0}^{i}, Z^{i}} \mathbb{E}[X_{T}^{i} 1_{\{\tau \leq T\}} - Y_{T}^{i} 1_{\{\tau > T\}}], & \text{as } l_0 = i
\end{cases}
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MFG among Principals

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Each Principal faces the optimization:

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V_0^P(p) = \max_{Y_0, Z} \mathbb{E}\left[ \int_0^T \lambda_t \Lambda_t X_t dt - \Lambda_T Y_T \right]
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where \( \lambda_t = \int (y - Y_t)_+ p_t(dy) \) and \( \Lambda_t = e^{-\int_0^t \lambda_s ds} \).
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Given \( \{p_t\} \), consider the contract represented by \( Y_0, Z \):

\[
\begin{align*}
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MFG among Principals

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Once \( \mathcal{L}(Y_t^*) = p_t \), the Principals reach a MFE.

**Proposition (Hu, R., Yang)**

*Further assume \( |Z|_\infty \leq C \). Then there exists such MFE.*
Conclusion

- The probabilist analysis on MFG is well compatible to the dynamic programming approach of PA problem
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- The probabilist analysis on MFG is well compatible to the dynamic programming approach of PA problem
- The PA perspective provides a probabilist model to the relaxed MF planning problem, and gives rise to some advantages
- The MFG formulation helps to solve the infinite-Principal case, by avoiding the difficult feature in the finite-Principal case
Thank you for your attention!