Mean Field Game in Principal-Agent Problem

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Séminaire de FIME
February 22, 2019

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$$\text{where} \quad dX_t^i = b(t, X_t^i, \mu_t^n, a_t^i)dt + \sigma(t, X_t^i, \mu_t^n, a_t^i)dW_t^i$$

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What is Mean Field Equilibrium?

A deterministic measure flow $(p_t)_{t\le T}$ is a MFE if

$$\begin{cases} a^* = \mathrm{argmax}_a \mathbb{E}\Big[\xi(X^a, p) - \int_0^T c(t, X_t^a, p_t, a_t)dt\Big] \\ p_t = \mathcal{L}(X_t^{a^*}) \end{cases}$$

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The MFE can be characterized as a fixed point of a PDE system or of a probabilistic model. For example, Carmona and Lacker (15') identified it as the solution of

$$\begin{cases} dY_t = -H(t, W_t, Z_t, p_t)dt + Z_t dW_t, \quad Y_T = \xi(W, p) \\ a_t^*(W) = \nabla_z H(t, W_t, Z_t, p_t) \\ p_t = \mathcal{L}(X_t^{a^*}) \end{cases}$$

where $H(t, x, z, p) = \max_{a \in \mathbb{R}}\{az - c(t, x, p, a)\}$.

Principal-Agent Problem with Moral Hazard

Optimal contracting between two parties, when Agent's effort cannot be observed, is a classical problem in Microeconomics, so-called Principal-Agent problem with moral hazard. It has applications in many areas of economics and finance, for example in corporate governance, portfolio management and energy transition.

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\text{Agent solves} \qquad & V_0^A = \max_a \mathbb{E}\Big[U\Big(\xi(X^a) - \int_0^T c(t, X_t^a, a_t)dt\Big)\Big] \\
& \text{where} \quad dX_t^a = a_t dt + dW_t \\
\text{Principal solves} \qquad & V_0^P = \max_\xi \mathbb{E}\Big[\tilde{U}\Big(X^{a^*}, \xi(X^{a^*})\Big)\Big]
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Mechanism: Principal plays first (giving ξ) and then Agent follows (responding by a^*). Though Principal CANNOT observe a, she CAN predict Agent's optimal rational response $a^*(\xi)$.

Dynamic programming approach to the PA problem

Assume Agent is risk-neutral, i.e. $U = I$. Agent's value function V^A and his best response a^* can be represented by the solution to the BSDE:

$$dY_t = -H(t, W_t, Z_t)dt + Z_t dW_t, \quad Y_T = \xi(W)$$
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$$\xi \in \Xi := \Big\{ Y_T = Y_0 - \int_0^T H(t, W_t, Z_t)dt + Z_t dW_t : \; Y_0 \in \mathbb{R}, Z \in \mathrm{BMO}_2 \Big\},$$

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the best response of Agent will be $a_t^* = \nabla_z H(t, W_t, Z_t)$.
Therefore, we can use (Y_0, Z) to represent both ξ and $a^*(\xi)$.

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Given the previous representation, we have

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This approach was pioneered by Sannikov (07') and Cvitanić, Possamaï & Touzi (15').

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Same idea... (see Elie, Mastrolia, Possamai 16')

Consider the contracts in the form:

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which is a classical McKean-Vlasov control problem.

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See e.g. *Porretta 14', Orrieri, Porretta, Savaré 18', Graber, Mészáros, Silva, Tonon 18', Benamou, Carlier, Di Marino, Nena 18'.*

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In order to solve this PA problem, we relax the contract ξ to be a path-dependent function.

Advantage of the relaxation

As before, we solve the PA problem by representing $(\xi, a^*(\xi), p)$ by (Y_0, Z)

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Note that (2) is a McKean-Vlasov SDE. In particular, if $H(t,x,z,p) = h(t,x,z) + f(t,x,p)$, then

$$dX_t = \nabla_z h(t, X_t, Z_t)dt + dW_t$$

is a simpler SDE, which admits weak solution under general conditions.

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Some extensions

Here are some feasible extensions:

- Let the volatility be controlled: $dX_t = a_t dt + \sigma_t dW_t$
- Since $\{\xi : p_T = \mu_1\} \neq \emptyset$, Principal can further study the optimal transport problem:

$$V_0^P = \max_{\xi : p_T = \mu_1} \mathbb{E}\big[\tilde{U}(X^{a^*}, \xi, p)\big].$$

This is a generalization to the (semi-)martingale transport problem.

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- Let I be a Poisson point proc. s.t. the first jump time τ follows the conditional law $\mathbb{P}[\tau \geq t | \mathcal{F}_t^{X,I}] = e^{-\int_0^t \lambda_s ds}$. The intensity proc. λ describes the hesitation of changing employer. The bigger λ is, the less hesitation Agent has. We shall allow Agent to control I through choosing λ.

Agent's problem: optimal switching

Given two contracts ξ^0, ξ^1, Agent solves the optimal switching problem

$$V_0^A = \max_{a,\lambda} \mathbb{E}\Big[\xi^0 1_{\{I_T=0\}} + \xi^1 1_{\{I_T=1\}} - \int_0^T \Big(c(t, X_t, a_t^{I_t}, I_t) + \frac{1}{2}|\lambda_t^{I_t}|^2\Big) dt\Big]$$

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Principal problem: Time Inconsistency

Based on the previous calculus, as before we may represent the contracts (ξ^0, ξ^1) by (Y_0^0, Y_0^1, Z^0, Z^1), i.e. consider the contracts in

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However, since Z does NOT dependent on I, this optimization is time-inconsistent.

More principals...

Consider the same model but with n Principals. After an appropriate normalization, the Agent's problem can be characterized by the system of BSDE:

$$dY_t^i = -\Big(\frac{1}{2(n-1)}\sum_{j\neq i}(Y_t^j - Y_t^i)_+^2 + Z_t^i\Big)dt + Z_t^i dW_t,\ \ Y_T^0 = \xi^i,\ 1 \leq i \leq n,$$

and the optimal intensity of switching to j-th Principal is $\lambda_t^{j,\star} = \frac{(Y^j - Y^i)_+}{n-1}$.

Let $n \to \infty$. Heuristically, the equation converges to

$$dY_t = -\Big(\frac{1}{2}\int (y - Y_t)_+^2 p_t(dy) + Z_t\Big)dt + Z_t dW_t, \quad p_t = \mathcal{L}(Y_t)$$

and once Agent leaves a company, there is NO chance he comes back.

MFG among Principals

Given $\{p_t\}$, consider the contract represented by Y_0, Z:

$$\xi \in \Xi(p) = \Big\{ Y_0 - \int_0^T \Big(\frac{1}{2}\int (y - Y_t)_+^2 p_t(dy) + Z_t\Big)dt + \int_0^T Z_t dW_t \Big\}$$

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Once $\mathcal{L}(Y_t^*) = p_t$, the Principals reach a MFE.

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$$\xi \in \Xi(p) = \Big\{ Y_0 - \int_0^T \Big(\frac{1}{2}\int (y - Y_t)_+^2 p_t(dy) + Z_t\Big) dt + \int_0^T Z_t dW_t \Big\}$$

Each Principal faces the optimization:

$$V_0^P(p) \;=\; \max_{Y_0, Z} \mathbb{E}\Big[\int_0^T \lambda_t \Lambda_t X_t dt - \Lambda_T Y_T\Big]$$

where $\lambda_t = \int (y - Y_t)_+ p_t(dy)$ and $\Lambda_t = e^{-\int_0^t \lambda_s ds}$.

Once $\mathcal{L}(Y_t^*) = p_t$, the Principals reach a MFE.

Proposition (Hu, R., Yang)

Further assume $|Z|_\infty \le C$. Then there exists such MFE.

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- The PA perspective provides a probabilist model to the relaxed MF planning problem, and gives rise to some advantages
- The MFG formulation helps to solve the infinite-Principal case, by avoiding the difficult feature in the finite-Principal case

Thank you for your attention!