Schauder Estimates for a Class of Potential Mean Field Games of Controls

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What is a mean field game?

Model for the following situation:

- **Infinitely many agents**, all identical and solving an optimal control problem
- They agents **interact**: the cost function of one single agent is influenced by all the others
- The agents do not cooperate $\longrightarrow$ **Nash equilibrium**.

The game is described by a coupled system of two PDEs:

1. **Hamilton-Jacobi-Bellman** (HJB) equation, describing the optimal behavior of each agent
2. **Fokker-Planck** equation (FP), describing the evolution of the distribution of the agents.

Goal

In this talk: proof of existence of a classical solution for the following **Mean Field Game of Controls**:

$$\begin{cases} (i) & -\partial_t u - \sigma \Delta u + H(\nabla u + P) = 0 & (x,t) \in Q, \\ (ii) & \partial_t m - \sigma \Delta m + \mathrm{div}(vm) = 0 & (x,t) \in Q, \\ (iii) & P(t) = \Psi\left(\int_{\mathbb{T}^d} v(x,t) m(x,t)\, \mathrm{d}x\right) & t \in [0,T], \\ (iv) & v = -\nabla H(\nabla u + P) & (x,t) \in Q, \\ (v) & m(x,0) = m_0(x), \quad u(x,T) = g(x) & x \in \mathbb{T}^d, \end{cases} \qquad \text{(MFGC)}$$

Specificity: coupling via the variable P (modelling a **price**), depending on both the distribution of the agents and **their controls**.

Nash equilibria

Consider the following situation with 2 agents making decisions $x_1 \in X_1$ and $x_2 \in X_2$ respectively:

- Agent 1 aims at minimizing $f_1(\cdot, x_2)$, when agent 2 plays x_2
- Agent 2 aims at minimizing $f_2(x_1, \cdot)$, when agent 1 plays x_1.

Definition

A pair $(\bar{x}_1, \bar{x}_2)$ is called **Nash equilibrium** if

$$\bar{x}_1 \in \underset{x_1 \in X_1}{\arg\min}\ f_1(x_1, \bar{x}_2) \quad \text{and} \quad \bar{x}_2 \in \underset{x_2 \in X_2}{\arg\min}\ f_2(\bar{x}_1, x_2).$$

Remark. Concept easily generalized to N agents.

Nash equilibria

Underlying assumptions.

- **Simultaneous** decisions.
- The agents **do not cooperate**. In some situations, they should: a pair (x_1, x_2) may exist, such that

$$f_1(\bar{x}_1, \bar{x}_2) > f_1(x_1, x_2) \quad \text{and} \quad f_2(\bar{x}_1, \bar{x}_2) > f_2(x_1, x_2).$$

- Agent 1 knows f_2, Agent 2 knows f_1.
  $\rightarrow$ Alternative (learning procedure): the game is repeated many times, and Agent 1 (resp. Agent 2) makes a prediction on the behavior of Agent 2 (resp. Agent 1):

$$x_1^k \in \underset{x_1 \in X_1}{\arg\min}\ f_1\Big(x_1, \tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} x_2^i\Big), \quad x_2^k \in \underset{x_2 \in X_2}{\arg\min}\ f_2\Big(\tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} x_1^i, x_2\Big).$$

Nash equilibria

An example of a game.

Consider N producers, buy some raw material on a market.

- Quantity bought by producer i: v_i
- Benefit resulting from v_i: $-L_i(v_i)$
- Unitary price of raw material: $P = \Psi(\sum_{i=1}^N v_i)$.
- Nash equilibrium: a vector $\bar{v} \in \mathbb{R}^N$ such that

$$\bar{v}_i \in \underset{v_i \in \mathbb{R}}{\arg\min} \left\{ L_i(v_i) + \Psi\left(\sum_{j=1}^N \bar{v}_j\right) v_i \right\},$$

for $i = 1, ..., N$.

Remark

The producers do not take into account their contribution to the equilibrium price P.

Assumptions:

- $L_1,\dots,L_N$ are strongly convex
- $\Psi = \nabla\Phi$, with Φ convex

Potential formulation:
Let $B\colon v \in \mathbb{R}^N \mapsto B(v) = \sum_{i=1}^N L_i(v_i) + \Phi(\sum_{i=1}^N v_i)$. Then,

$$\bar{v} \in \mathbb{R}^N \text{ is a Nash equilibrium}$$
$$\Longleftrightarrow \underbrace{\nabla L_i(\bar{v}_i) + \Psi\left(\textstyle\sum_{j=1}^N \bar{v}_j\right)}_{=\nabla_{v_i} B(\bar{v})} = 0, \ \forall i = 1, \dots, N$$
$$\Longleftrightarrow \bar{v} \text{ minimizes } B.$$

The mapping B is strongly convex, thus there exists a unique Nash equilibrium.

Nash equilibria

Remark

Our MFG of controls is a **dynamical** *version of the situation described above, with a continuum of agents.*

Reformulation of the equilibrium conditions.
Convex conjugate of L_i:

$$L_i^*(\lambda) = \sup_{v_i \in \mathbb{R}} \lambda v_i - L_i(v_i).$$

Since L_i is strongly convex, L_i^* is differentiable, moreover,

$$L_i^*(\lambda) = \langle \lambda, v_i \rangle - L_i(v_i) \Longleftrightarrow v_i = \nabla L_i^*(\lambda).$$

Therefore, $\bar{v}$ is a Nash equilibrium if and only if

$$\bar{v}_i = \nabla L_i^*(-P), \; \forall i = 1, ..., N \quad \text{and} \quad P = \Psi(\textstyle\sum_{i=1}^N \bar{v}_i).$$

HJB equation

Consider the following **stochastic optimal control problem**:

$$u(x,t) = \inf_{V \in \mathbb{L}^2(t,T)} \mathbb{E}\left[\int_t^T L(V(s)) + \langle P(s), V(s) \rangle \, \mathrm{d}s + g(X(T)) \right],$$

$$\text{subject to: } \begin{cases} \dot{X}(s) = & V(s) + \sqrt{2\sigma} W(s), \;\; s \in (t,T) \\ X(t) = & x, \end{cases} \qquad (OCP)$$

given $L \colon \mathbb{R}^d \to \mathbb{R}$, $P \in L^2(0,T;\mathbb{R}^d)$, $g \colon \mathbb{R}^d \to \mathbb{R}$, and $x_0 \in \mathbb{R}^d$.

Application: charge of an electrical vehicle.

- speed of charge (at time s): $V(s)$
- unitary price of electricity: $P(s)$
- state of charge of the battery: $X(s)$.

HJB equation

Assume that g is periodic with period 1. Let $Q = \mathbb{T}^d \times (0, T)$. Let $H(p) = L^*(-p)$. The value function is a viscosity solution to

$$\begin{cases} -\partial_t u - \sigma \Delta u = -H(P(t) + \nabla u(x,t)), & (x,t) \in Q, \\ u(x,T) = g(x), & x \in \mathbb{T}^d, \end{cases} \qquad \text{(HJB)}$$

For a solution $\bar{V}$ with associated trajectory $\bar{X}$, we have:

$$\bar{V}(t) = -\nabla H(\nabla u(\bar{X}(t), t) + P(t)) =: v(\bar{X}(t), t).$$

Remark

The **optimal feedback law** $v(x,t) = -\nabla H(\nabla u(x,t)) + P(t)$ *does not depend on the initial condition of (OCP).*

Fokker-Planck equation

Back to the Mean Field Game model.

- Continuum of identical agents (with different initial conditions), all solving (OCP), thus all using the same feedback law.
- Let m denote the **distribution** of the agents:

$$\int_{\omega} m(x,t)\,\mathrm{d}x \rightarrow \text{Proportion of agents located in } \omega \text{ at time } t.$$

- The distribution m is solution to the Fokker-Planck equation:

$$\begin{cases} \partial_t m - \sigma \Delta m + \operatorname{div}(vm) = 0, & (x,t) \in Q, \\ m(x,0) = m_0(x), & x \in \mathbb{T}^d, \end{cases} \qquad \text{(FP)}$$

where the initial distribution m_0 is given.

Mean Field Game of Controls

Complete model:

$$\begin{cases} (i) & -\partial_t u - \sigma\Delta u + H(\nabla u + P) = 0 & (x,t) \in Q, \\ (ii) & \partial_t m - \sigma\Delta m + \operatorname{div}(vm) = 0 & (x,t) \in Q, \\ (iii) & P(t) = \Psi\left(\int_{\mathbb{T}^d} v(x,t)m(x,t)\,\mathrm{d}x\right) & t \in [0,T], \\ (iv) & v = -\nabla H(\nabla u + P) & (x,t) \in Q, \\ (v) & m(x,0) = m_0(x), \quad u(x,T) = g(x) & x \in \mathbb{T}^d, \end{cases} \qquad \text{(MFGC)}$$

Unknown: $u = u(x,t)$, $m = m(x,t)$, $P = P(t)$, $v = v(x,t)$.
Endogenous price P (as in the introductory example).

Remark

If P is exogenous and (iii) removed, then the system is decoupled.

Application example: car drivers buy electricy on a small market.

Functional framework

Given $\alpha \in (0,1)$ and $X = [0,T]$, $X = \mathbb{T}^d$, or $X = Q$,

$$
\begin{aligned}
C^{j+\alpha}(X) :=& \big\{ u \in C^j(X) \,|\, \exists C > 0,\ \forall x, y \in X, \\
& \|D^i u(y) - D^i u(x)\| \leq C\|y - x\|_X^\alpha, \text{whenever } |i| \leq j \big\}, \\
C^{\alpha,\alpha/2}(Q) :=& \big\{ u \in C(Q) \,|\, \exists C > 0,\ \forall x, y \in X, \\
& |u(x_2,t_2) - u(x_1,t_1)| \leq C\big(\|x_2 - x_1\|^\alpha + |t_2 - t_1|^{\alpha/2}\big) \big\} \\
C^{2+\alpha,1+\alpha/2}(Q) :=& \big\{ u \in C^{\alpha,\alpha/2}(Q) \,|\, \partial_t u \in C^{\alpha,\alpha/2}(Q), \\
& \nabla u \in C^{\alpha,\alpha/2}(Q),\ \nabla^2 u \in C^{\alpha,\alpha/2}(Q) \big\}.
\end{aligned}
$$

We fix $p > d+2$ and define the Sobolev space

$$W^{2,1,p}(Q) := L^p(0,T; W^{2,p}(Q)) \cap W^{1,p}(Q).$$

Embedding: $\|u\|_{C^\alpha(Q)} + \|\nabla u\|_{C^\alpha(Q)} \leq C\|u\|_{W^{2,1,p}(Q)}$.

Assumptions

Monotonicity assumptions:

- $\Psi = \nabla\Phi$, where Φ is convex
- L is strongly convex.

Growth assumptions:

- $L(v) \leq C(1 + \|v\|^2)$
- $\Psi(z) \leq C(1 + \|z\|)$.

Regularity assumptions:

- $H \in C^2(\mathbb{R}^d)$, H, ∇H, $\nabla^2 H$ are locally Hölder continuous
- Ψ is locally Hölder continuous
- $m_0 \in C^{2+\alpha}(\mathbb{T}^d)$, $g \in C^{2+\alpha}(\mathbb{T}^d)$
- $m_0 \in \mathcal{D}_1(\mathbb{T}^d) := \{h \in L^\infty(\mathbb{T}^d) \,|\, h \geq 0,\ \int_{\mathbb{T}^d} h(x)\,dx = 1\}$.

Auxiliary mappings

We analyse (*iii*) and (*iv*) to eliminate v and P from (MFGC).

Lemma

For all $m \in \mathcal{D}_1(\mathbb{T}^d)$, for all $w \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$, there exists a unique pair $(v, P) = (\mathbf{v}(m, w), \mathbf{P}(m, w)) \in L^\infty(\mathbb{T}^d, \mathbb{R}^d) \times \mathbb{R}^d$ such that

$$\begin{cases} v(x) = & -\nabla H(w(x) + P), \quad \forall x \in \mathbb{T}^d, \\ P = & \Psi\left(\int_{\mathbb{T}^d} v(x) m(x)\, dx\right). \end{cases} \qquad (*)$$

Elements of proof. If $m > 0$, then (v, P) satisfies $(*)$ if and only if v minimizes the following convex functional:

$$J(v) \colon v \mapsto \Phi\left(\int_{\mathbb{T}^d} v(x) m(x)\, \mathrm{d}x\right) + \int_{\mathbb{T}^d} \big(L(v(x)) + \langle w(x), v(x) \rangle\big) m(x)\, \mathrm{d}x,$$

which possesses a unique minimizer.

Auxiliary mappings

Reduced coupled system:

$$\begin{cases} -\partial_t u - \sigma\Delta u + H(\nabla u + \mathbf{P}(m(\cdot,t), \nabla u(\cdot,t))) = 0, \\ \partial_t m - \sigma\Delta m + \operatorname{div}(\mathbf{v}(m(\cdot,t), \nabla u(\cdot,t))m) = 0, \\ u(x,T) = g(x), \quad m(x,0) = m_0(x). \end{cases} \qquad (MFGC')$$

Lemma (Stability lemma)

Let $R > 0$, let m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$, let w_1 and $w_2 \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$ with $\|w_i\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq R$. There exists $C > 0$ and $\alpha \in (0,1)$, depending on R only such that

$$\begin{aligned} &\|\mathbf{P}(m_2, w_2) - \mathbf{P}(m_1, w_1)\| \\ &\quad \leq C\left(\|w_2 - w_1\|_{L^\infty(\mathbb{T}^d)}^\alpha + \|m_2 - m_1\|_{L^1(\mathbb{T}^d)}^\alpha\right). \end{aligned}$$

Idea of proof: stability analysis for convex optimization problems.

Potential formulation

Consider the cost function $B\colon W^{2,1,p}(Q) \times L^\infty(Q) \to \mathbb{R}$,

$$B(m,v) = \iint_Q L(v(x,t))m(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)m(x,T)\,\mathrm{d}x + \int_0^T \Phi\Big(\int_{\mathbb{T}^d} v(x,t)m(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t.$$

Lemma

Let $(\bar{u},\bar{m},\bar{v},\bar{P}) \in W^{2,1,p}(Q)^2 \times L^\infty(Q,\mathbb{R}^d) \times L^\infty(0,T;\mathbb{R}^k)$ be a solution to $(MFGC)$. Then, $(\bar{m},\bar{v})$ is a **solution** *to:*

$$\min_{\substack{m\in W^{2,1,p}(Q)\\ v\in L^\infty(Q,\mathbb{R}^k)}} B(m,v) \quad s.t.: \begin{cases} \partial_t m - \sigma\Delta m + div(vm) = 0, \\ m(x,0) = m_0(x). \end{cases} \qquad (\mathcal{P})$$

Result and approach

Theorem

There exists a classical solution to (MFGC) with

$$\begin{aligned} &u \in C^{2+\alpha,1+\alpha/2}(Q), && m \in C^{2+\alpha,1+\alpha/2}(Q), \\ &v \in C^{\alpha}(Q), D_x v \in C^{\alpha}(Q), && P \in C^{\alpha}(0,T). \end{aligned}$$

Theorem (Leray-Schauder)

Let X be a Banach space and let $\mathcal{T} \colon X \times [0,1] \to X$ satisfy:

1. *$\mathcal{T}$ is a continuous and compact mapping,*
2. $\exists \tilde{x} \in X$, $\mathcal{T}(x,0) = \tilde{x}$ *for all* $x \in X$,
3. $\exists C > 0$, $\forall (x,\tau) \in X \times [0,1]$,

$$\mathcal{T}(x,\tau) = x \Longrightarrow \|x\|_X \leq C.$$

Then, there exists $x \in X$ such that $\mathcal{T}(x,1) = x$.

Parabolic estimates

Consider the parabolic equation:

$$\begin{cases} \partial_t u - \sigma\Delta u + \langle b, \nabla u\rangle + cu = & h, \qquad (x,t) \in Q, \\ u(x,0) = & u_0(x), \quad x \in \mathbb{T}^d. \end{cases}$$

Assume that $u_0 \in C^{2+\alpha}(\mathbb{T}^d)$.

Theorem

1. *Assume that* $b \in L^p(Q)$, $c \in L^p(Q)$, *and* $h \in L^p(Q)$. *Then,* $u \in W^{2,1,p}(Q)$, $u \in C^\alpha(Q)$, *and* $\nabla u \in C^\alpha(Q)$.
2. *Assume that* $b \in C^{\beta,\beta/2}(Q)$, $c \in C^{\beta,\beta/2}(Q)$, *and* $h \in C^{\beta,\beta/2}(Q)$. *Then,* $u \in C^{2+\alpha,1+\alpha/2}(Q)$.

Construction of $\mathcal{T}$

Let $X = (W^{2,1,p}(Q))^2$. For $(u, m, \tau) \in X \times [0,1]$,
$(\tilde{u}, \tilde{m}) = \mathcal{T}(u, m, \tau) \in W^{2,1,p}(Q)^2$ where:

- $\tilde{u}$ is the solution to

$$\begin{cases} -\partial_t \tilde{u} - \sigma\Delta\tilde{u} + \tau H(\nabla u + \mathbf{P}(\rho(m), \nabla u)) = & 0, \\ \tilde{u}(T, x) = & \tau g(x), \end{cases}$$

- $\tilde{m}$ is the solution

$$\begin{cases} \partial_t \tilde{m} - \sigma\Delta\tilde{m} + \tau \mathrm{div}(\mathbf{v}(\rho(m), \nabla u) m) = & 0, \\ \tilde{m}(x, 0) = & m_0(x), \end{cases}$$

Here $\rho\colon L^\infty(\mathbb{T}^d) \to \mathcal{D}_1(\mathbb{T}^d)$ is a kind of regular projection operator $(\rho(m) = m$ for $m \in \mathcal{D}_1)$.

Regularity of $\mathcal{T}$

Lemma

1. *The mapping $\mathcal{T}$ is continuous.*
2. *For all $R > 0$, there exist $C > 0$ and $\alpha \in (0,1]$ such that for all $(u,m) \in W^{2,1,p}(Q)$ and for all $\tau \in [0,1]$,*

$$\|u\|_{W^{2,1,p}(Q)} + \|m\|_{W^{2,1,p}(Q)} \leq R \\ \implies \|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(Q)} + \|\tilde{m}\|_{C^{(2+\alpha,1+\alpha/2}(Q)} \leq C,$$

where $(\tilde{u}, \tilde{m}) = \mathcal{T}(u,m,\tau)$.

Consequence: $\mathcal{T}$ is compact, by the theorem of Arzelà-Ascoli.

Estimates for fixed points

Proposition

There exist $C > 0$ *and* $\alpha \in (0,1)$ *such that for all* $(u, m, \tau) \in X \times [0,1]$ *satisfying* $(u,m) = \mathcal{T}(u,m,\tau)$, *we have*

$$\begin{aligned} &\|u\|_{C^{2+\alpha,1+\alpha/2}(Q)} \leq C, && \|m\|_{C^{2+\alpha,1+\alpha/2}(Q)} \leq C, \\ &\|v\|_{C^{\alpha}(Q)} + \|D_x v\|_{C^{\alpha}(Q)} \leq C, && \|P\|_{C^{\alpha}(0,T)} \leq C, \end{aligned}$$

where $P = \mathbf{P}(m, \nabla u)$ *and* $v = \mathbf{v}(m, \nabla u)$.

Proof. For $\tau = 1$. The pair (m, v) is a solution to $(\mathcal{P})$. Thus,

$$C \iint_Q \|v(x,t)\|^2 m(x,t)\, \mathrm{d}x\, \mathrm{d}t - C \leq B(m,v) \leq B(m_0, v_0 = 0) \leq C.$$

Thus,

$$\|P\|^2_{L^2(0,T)} \leq C\Big(1 + \int_0^T \|\textstyle\int_{\mathbb{T}^d} vm\, \mathrm{d}x\|^2 \mathrm{d}t\Big) \leq C\Big(1 + \iint_Q \|v\|^2 m\, \mathrm{d}x\, \mathrm{d}t\Big) \leq C.$$

Estimates for fixed points

$u, \nabla u \in L^\infty(Q)$	u value function of opt. control pb.
$P \in L^\infty(0, T; \mathbb{R}^k)$	Stability lemma
$H(\nabla u + P) \in L^\infty(Q)$ $u \in W^{2,1,p}(Q)$	Regularity of H HJB: parabolic eq. with L^p coeff.
$v \in L^\infty(Q, \mathbb{R}^d)$ $D_x v \in L^p(Q, \mathbb{R}^{d\times d})$	Stability lemma $D_x v = -\nabla^2 H(\nabla u + P)\nabla^2 u$
$m \in W^{2,1,p}(Q)$	FP: parabolic eq. with L^p coeff.
$P \in C^\alpha(Q)$	Stability lemma
$H(\nabla u + P) \in C^\alpha(Q)$ $u \in C^{2+\alpha,1+\alpha/2}(Q)$	Regularity of H HJB: parabolic eq. with Hölder coeff.
$v, D_x v \in C^\alpha(Q)$	Stability lemma
$m \in C^\alpha(Q)$	FP: parabolic eq. with Hölder coeff.

Duality

Consider the following criterion:

$$D(u,P) = -\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x - \int_0^T \Phi^*(P(t))\,\mathrm{d}t,$$

for $u \in W^{2,1,p}(Q)$ and $P \in L^\infty(0,T)$, and the **dual** problem:

$$\sup_{\substack{u\in W^{2,1,p}(Q)\\ P\in L^\infty(0,T)}} D(u,P), \quad \text{s.t.:} \begin{cases} -\partial u_t - \sigma\Delta u + H(\nabla u + P) = 0, \\ u(x,T) = g(x). \end{cases}$$

Lemma

For all solutions $(\bar{u}, \bar{m}, \bar{v}, \bar{P})$ to (MFGC), the pair $(\bar{u}, \bar{P})$ is a **solution** *to the dual problem.*

Duality

Let (u, P) be feasible.

$$\begin{aligned}\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} u(x,T)\bar{m}(x,T)\,\mathrm{d}x \\ &= -\iint_Q \partial_t u\bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q u\partial_t\bar{m}\,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q (\sigma\Delta u - H(\nabla u + P))\bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q u(\sigma\Delta\bar{m} - \mathrm{div}(\bar{v}\bar{m}))\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \iint_Q \big(L(\bar{v}) + \langle\nabla u + P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t - \iint_Q \langle\nabla u, \bar{v}\rangle\bar{m}\,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q \big(L(\bar{v}) + \langle P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Therefore,

$$\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x + \iint_Q \big(L(\bar{v}) + \langle P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t.$$

Duality

We also have:

$$-\int_0^T \Phi^*(P(t))\,\mathrm{d}t \le -\int_0^T \left\langle P(t), \textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\,\mathrm{d}x\right\rangle + \int_0^T \Phi\big(\textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\big)\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned} D(u,P) \le \; & \iint_Q L(\bar{v})\bar{m}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x \\ & + \int_0^T \Phi\big(\textstyle\int_{\mathbb{T}^d} \bar{v}\bar{m}\big)\,\mathrm{d}t = B(\bar{v},\bar{m}). \end{aligned}$$

Equality holds for $(u,P) = (\bar{u},\bar{P})$.

Outlook

Summary:

- Existence result for a $(MFGC)$ based on a fixed-point theorem.
- A priori estimates for fixed points obtained with the help of a potential formulation.

Additional results:

- Uniqueness.
- HJB equations of the following form, with f smooth:

$$-\partial_t u - \sigma \Delta u + H(\nabla u + P) = f(m(\cdot, t)).$$

- H can depend on (x, t), Ψ can depend on t.

Future work:

- Convergence of a learning procedure.

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Thank you for your attention!